SOME ENERGY - RELATED FUNCTIONALS, AND THEIR VERTICAL VARIATIONAL THEORY

by

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Dedicated to my sister, Rosie.

In memory of our brother, Tim.

" Lord, you establish peace for me;

all that I have accomplished, you have done for me."

Isaiah 26; 12.

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SUMMARY

The energy and volume of a mapping of Riemannian manifolds are linked by a discrete family of functionals, indexed by the elementary symmetric polynomials. We explore the variational properties of members of this family; in particular, their tension fields, stress-energy tensors, and Jacobi operators.

When one Riemannian manifold fibres over another, applying the conventional theory of harmonic maps to sections neglects the additional structure supplied by the fibering. We give an alternative definition for harmonicity of sections which overcomes this deficiency, and is closely- enough linked to the conventional theory to share many of the qualitative properties of harmonic maps. Such "harmonic sections" arise as solutions to a variational problem, one consequence of which is to allow the proof of a reduction theorem for harmonic sections of a Riemannian vector bundle.

The "Gauss section" of an isometrically immersed submanifold extends the idea of "Gauss map" to ambient spaces of arbitrary curvature. We prove a non-trivial identity relating Gauss map to second fundamental form, and generalize to the Gauss section. This leads to a characterization of immersions with harmonic Gauss section, and to a further identity involving a suitably generalized notion of "third fundamental form". We are also able to characterize those Riemannian foliations of codimension one whose Gauss section is harmonic.

INTRODUCTION

Given an m-dimensional, smooth, oriented. Riemannian manifold (M,g), a symmetric function $\sigma:\mathbb{R}^m \to \mathbb{R}$, and a diagonalisable tensor field B on M of type $(1,1)$, we may form the σ-integral associated to B by evaluating σ on the spectrum of B and integrating over (M,g). This construction is mentioned at the beginning of [E-S], where it is also noted that when $\varphi: (M^m,g) \to (N^n,h)$ is a mapping of Riemannian manifolds:-

(i) The energy of φ is the $\sigma_1/2$- integral,

(ii) The volume of φ is closely related to the σ_m-integral,

where the σ_r-integral in question is that associated to the $(1,1)$ tensor obtained (by "raising an index") from the pulled-back metric φ^*h, and the r^{th} elementary symmetric polynomial σ_r. Since then, R.Reilly has worked-out the variational theory for the σ_r-integrals associated to the shape operator of an immersed hypersurface ([Rei]), fundamental to the mechanics of which are certain divergence-free Newton tensors. Inspired by his example, and undeterred by the fact that the relevant Newton tensors are no longer divergence-free, Chapter 1 of this thesis attempts to do the same for the family of energy-related σ_r-integrals. We begin by laying out with some care the tools of the variational trade - for these are to be redeployed in Chapter 3 where slightly more subtlety is involved - and proceed to investigate the critical points of σ_r under variations of the two independent parameters of map φ, and domain metric g. In analogy with the theory for σ_1, we have called the corresponding Euler-Lagrange operators higher-power tension fields, and stress-energy tensors, respectively. The qualitative picture is much as one would expect. For instance, on ascending the ladder (whose bottom and top rungs are the energy and volume functionals respectively), the class of "trivially r-harmonic" mappings widens to admit those which fail to attain successively higher rank. This is reflected in the quasi-linearity of the higher-power tension fields and Jacobi operators, with consequent complications regarding their ellipticity. There is also a generalization to even-dimensional manifolds of the conformal invariance and volume-majorizing properties of the energy functional on maps of surfaces - a natural idea here is that of <u>r-conformality</u> of a map, generalizing weak conformality. On account of this, it is tempting to look for nice links between properties of r-harmonic maps of 2r-manifolds, and the complex geometry of any holomorphic structures which may be imposable. However, there is evidence to suggest that the "bottom rung" may be somewhat special in this respect; notably that, when r 1, any r-conformal, r-harmonic map of a 2r-manifold is necessarily homothetic (or r-trivial) (Propositions 1.7.2 and 1.7.3)! No such thing is true for weakly conformal harmonic maps of a surface, although in this case much is owed to the remarkable properties of Riemann surfaces. We conclude Chapter 1 with a calculation of the Euler-Lagrange equations for the volume functional, making use of higher-power stress-energy.

Chapter 2 is a catalogue of various results in the geometry of fibre bundles, for later use, but recorded here to establish notation and allow the expositions of later results to proceed unhindered. Sections 3 and 4, which are referred to in Chapter 4, deal with the geometry of fibre products and endomorphism bundles, and are almost self-evident. So also is our brief account of connections in vector bundles from the viewpoint of <u>connection maps</u> (§6), which seems the most appropriate for the discussion of Chapter 3 §3. Section 2 is a useful decomposition result for curves in a bundle with connection, and is the lynchpin for the proof of the fundamental identity of Chapter 4 (Proposition 4.2.4). The section on <u>partial connections</u> (§5) contains a unified proof of two well-known results concerning Riemannian immersions and submersions. Besides being of interest in itself, this result is used throughout the thesis.

The title of Chapter 3 is perhaps a little misleading, for the idea of "harmonic sections" is to generalize the graph of a harmonic map rather than to look at bundle cross-sections which are themselves harmonic maps. Thus, we are interested not simply in sections whose tension field vanishes, but those with vanishing vertical tension field. In the first instance, the generalization from graphs to sections of non-trivial fibre bundles is made by looking at the associated equivariant maps. We then introduce the (higher-power) vertical energy functionals and examine their variational theory. It is not surprising that the two notions of harmonic section thus arising coincide. However, it is interesting to note that the apparently greater degree of generality permitted by the variational viewpoint (namely, loosening "fibre bundle" to any submersion of Riemannian manifolds, and "section" to any map) cannot be fully allowed if there is to be any hope of computing the Euler-Lagrange equations. Indeed, we are "forced" into considering the restricted variational problem of varying a section through sections, and placing (fairly natural) geometrical constraints on the submersion itself.

Having hitherto worked with the entire range of energy-related functionals, we now discard the higher-power energies and look at the 1-harmonic sections of vector bundles. From the variational point of view it is natural to consider Riemannian vector bundles (Lemma 3.3.3) and compactly-supported sections (more generally, sections of finite vertical energy), in which case linearity permits the reduction of the second order vertical tension field to a first order system. This is seen as a generalization of the fact that every harmonic map of finite energy into Euclidean space is (locally) constant. We conclude the

Chapter by rephrasing for harmonic sections the existence and unique continuation properties of harmonic maps.

In Chapter 4 we exploit the theory of harmonic sections to study the geometry of isometrically immersed submanifolds. When the ambient space is Euclidean, or more generally of constant curvature, it is possible to define an associated Gauss map, which takes its values in an appropriate Grassmannian. The geometry of the Gauss map is then nicely related to the geometry of the immersion (cf [R-V], [C-G], [Oba], [Ish]). By replacing the Gauss map with the Gauss section (of the appropriate Grassmann bundle), it is possible to obtain a complete generalization which places no restrictions on the ambient space, and has correspondingly richer geometrical content. At the heart of the matter is a connection-preserving vector bundle fibre isometry (!) which identifies the 2nd fundamental form of the immersion with the vertical differential of its Gauss section (Proposition 4.2.4). Having obtained this identity, we find ourselves on level ground. On one hand, applying the Codazzi equation gives an instant generalization of the theorem of Ruh-Vilms ; on the other, after making a suitable definition of the third fundamental form, an application of the Gauss equation gives an identity of quadratic differentials extending that of M.Obata ([R-V], [Oba]). Through the consideration of a couple of examples, we are lead to the idea of an Einsteinian immersion, in which case the generalized version of Ruh-Vilms theorem simplifies to a virtual facsimile of its original. Being fulfilled whenever the ambient space has constant curvature, "Einsteinian" is also the extra condition required for the theorems of Obata's paper to find their place in our more general setting (Corollary 4.2.6).

Preceding the generalization described above is a detailed discussion of the case when the ambient space is Euclidean, for which we offer the following three justifications. The first is simply to fill a gap in the literature, previous expositions (e.g. [R-V], [C-G]) having omitted to supply details of their implicit geometrical identifications (see Propositions 4.1.1, 4.1.3, and 4.1.4 below). However, when the ambient space is flat, it is in these very identifications that all the geometry lies! Secondly, the properties of one of the identifications (Proposition 4.1.3) are needed to prove the "vertical part" of its subsequent generalization (Proposition 4.2.3). And thirdly, the proof of Proposition 4.2.4 is directly motivated by that of Proposition 4.1.4.

Our original motivation for a generalization of Ruh-Vilms' theorem came from the study of the Gauss section of a <u>Riemannian foliation</u> - in particular, the question " When is the Gauss section harmonic?" (following a tentative conjecture appearing on the last page of [KT2]) - and the concluding section of Chapter 4 returns to this. Of course, the question is well-posed for foliations of arbitrary codimension, but in choosing to approach the problem by piecing together the Gauss sections of individual leaves, matters are considerably simplified in codimension one. The basic reason for this is that a codimension one Riemannian foliation is invariant under parallel translation along normal geodesics, with the consequence that there is no normal contribution to the vertical tension field of the full Gauss section. It also happens that in codimension one the generalized Ruh-Vilms theorem also simplifies, so that the final result (Theorem 4.3.1) is fairly succint. Even so, the explicit appearance of curvature shows the conjecture of [KT2] to be rather over-optimistic. As a particularly nice example, we cite the <u>isoparametric hypersurface</u> foliation of a space form (after removing focal varieties), whose Gauss section is always harmonic.

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NOTATION

For the most part, our notation agrees with that of Kobayashi and Nomizu ([K-N]), including their choice of sign for the curvature tensor. Points at which we may diverge are usually due to the influence of Spivak ([Spi]); for example:

If $\sigma:E \to M$ is a (smooth) fibre bundle, then the space of (smooth) sections of σ is denoted by $\mathfrak{C}(E)$.

If $\sigma:E \to M$ is a vector bundle, with dual bundle $\sigma^*:E^* \to M$, then we abbreviate the tensor bundle $\underbrace{E^* \otimes \ldots \otimes E^*}_{k} \otimes \underbrace{E \otimes \ldots \otimes E}_{\ell} \to M$ to $\mathfrak{T}^k_\ell(\sigma):\mathfrak{T}^k_\ell(E) \to M$, and write $\mathfrak{T}(E) = \bigoplus_{k,\ell=1}^{\infty} \mathfrak{T}^k_\ell(E)$. In particular, when $\sigma = \tau_M:TM \to M$, we abbreviate $\mathfrak{T}^k_\ell(TM)$ to $\mathfrak{T}^k_\ell(M)$.

The Lie algebra of a Lie group is, on the whole, denoted by the corresponding lower case Gothic character. For the classical groups, this is only partially true; for example, $O(n)$ has Lie algebra $\mathfrak{o}(n)$.

The summation convention is taken to be in force, unless otherwise indicated.

As regards cross-referencing, each Remark, Lemma, Proposition, Theorem and Corollary is numbered with respect to the section (§) in which it appears. If mentioned elsewhere in the same chapter, this number is preceded by the relevant section number; if referred to in another chapter, a further prefix is added.

CHAPTER 1 : A FAMILY OF ENERGY FUNCTIONALS

§1. PRODUCT MANIFOLDS

Let π_P (resp. π_Q) denote the projection from the product $P \times Q$ of two smooth manifolds onto P (resp. Q), and, for each $(x,y) \in P \times Q$, let

$$i_y : P \to P \times Q;\ x \mapsto (x,y),\ j_x : Q \to P \times Q;\ y \mapsto (x,y)$$

be the corresponding inclusions. The tangent bundles $\tau_P : TP \to P$ and $\tau_Q : TQ \to Q$ may be "spread out" over $P \times Q$ by defining vector bundles:

$$\bar{\tau}_P : \overline{TP} \to P \times Q;\ \overline{TP} = \coprod_{(x,y)} di_y(T_x P) \text{ etc.}$$

so that $T(P \times Q)$ is isomorphic to $\overline{TP} \oplus \overline{TQ}$. Any vector field on P (or Q) may then be extended to $P \times Q$; for example, if $X \in \mathfrak{C}(TP)$, define $\bar{X} \in \mathfrak{C}(\overline{TP})$ by $\bar{X} \circ i_y = di_y \circ X$. A *product field* on $P \times Q$ is any of the form $\bar{X} + \bar{Y}$, where $X \in \mathfrak{C}(TP)$ and $Y \in \mathfrak{C}(TQ)$; by writing (v,w) for $di_y(v) + dj_x(w) \in T_{(x,y)}(P \times Q)$ (where $v \in T_x P$ and $w \in T_y Q$) we have that $\bar{X} = (X,0)$, $\bar{Y} = (0,Y)$, and $\bar{X} + \bar{Y} = (X,Y)$. Conversely, starting with a vector field Z on $P \times Q$ and restricting to fibres of π_P (resp. π_Q) produces a family of vector fields on P (resp. Q):

$$\{Z_y = d\pi_P \circ Z \circ i_y;\ y \in Q\} \subset \mathfrak{C}(TP).$$

Say that Z is Q-*invariant* if $Z_{y_1} = Z_{y_2}$ for all $y_i \in Q$. Product fields are clearly both P-invariant and Q-invariant - they are the only such fields.

Through the existence of local product frame fields in $T(P \times Q)$, product fields are sufficient to characterise geometry in $P \times Q$. Thus, if (P,k) and (Q,ℓ) are Riemannian manifolds, $P \times Q$ has the *product metric:*

$$k \times \ell((X,Y),\ (X',Y')) = k(X,X') + \ell(Y,Y')$$

w.r.t. which $\overline{TP}$ and $\overline{TQ}$ are orthogonal, and each of π_P, π_Q a Riemannian submersion. Also:

Proposition

The Levi-Civita connection for $(P \times Q,\ k \times \ell)$ is characterised by:

$${}^{P\times Q}\nabla_{(X,Y)}(X',Y') = ({}^{P}\nabla_X X',\ {}^{Q}\nabla_Y Y')$$

for all $X, X' \in \mathcal{C}(TP)$ and $Y, Y' \in \mathcal{C}(TQ)$.

Proof. Such a connection in $T(P \times Q)$ is metric and torsion-free. $\square$

§2. INDUCED VECTOR BUNDLES

Let $\pi: E \to N$ be a vector bundle, and $\phi: P \to N$. Write $\phi^{-1}\pi : \phi^{-1}E \to P$ for the vector bundle induced by ϕ, and $\tilde{\phi}$ for the *pullback morphism:*

$$\phi^{-1}E = \{(p,e) \in P \times E;\ \pi(e) = \phi(p)\}$$

$$\begin{tikzcd} \phi^{-1}E \arrow[r, "\tilde{\phi}"] \arrow[d, "\phi^{-1}\pi"'] & E \arrow[d, "\pi"] \\ P \arrow[r, "\phi"'] & N \end{tikzcd} \quad ; \quad \tilde{\phi}(p,e) = e$$

Corresponding to any section α of π is the *pullback section* $\phi^{-1}\alpha$ of $\phi^{-1}\pi$:

$$\phi^{-1}\alpha(p) = (p,\alpha(\phi(p))).$$

Remark 1. If $\phi = \pi_P : P \times Q \to P$, then $\pi_P^{-1}(TP) \cong \overline{TP}$ and $\pi_P^{-1}(X) \leftrightarrow \bar{X}$ for any $X \in \mathfrak{C}(TP)$.

Since pullback sections provide local fibre bases for $\phi^{-1}\pi$, they are sufficient to characterise geometry in $\phi^{-1}E$. Thus, if π has fibre metric g, $\phi^{-1}\pi$ has the *pullback metric* $\phi^{-1}g$:

$$\phi^{-1}g(\phi^{-1}\alpha,\phi^{-1}\beta) = g(\alpha,\beta), \text{ for all } \alpha,\ \beta \in \mathfrak{C}(E).$$

Similarly, if π has connection ∇, $\phi^{-1}\pi$ has the *pullback connection* $\phi^{-1}\nabla$:

$$(\phi^{-1}\nabla)_Z(\phi^{-1}\alpha) = \phi^{-1}(\nabla_{d\phi(Z)}\alpha), \text{ for all } Z \in \mathfrak{C}(TP).$$

Remark 2. If (E,g,∇) is a *Riemannian* vector bundle, so is $(\phi^{-1}E,\phi^{-1}g,\phi^{-1}\nabla)$.

A morphism $\begin{tikzcd} TP \arrow[r, "A"] \arrow[d, "\tau_P"'] & E \arrow[d, "\pi"] \\ P \arrow[r, "\phi"'] & N \end{tikzcd}$ may be regarded as a section of $T^*P \underset{P}{\otimes} \phi^{-1}E$. The A-*torsion* of ∇ (see [KT2] is then that $\phi^{-1}E$-valued 2-form on P:

$$T_{\nabla,A}(Z,Z') = (\phi^{-1}\nabla)_Z(AZ') - (\phi^{-1}\nabla)_{Z'}(AZ)-A[Z,Z']$$

for all $Z,Z' \in \mathfrak{C}(TP)$. If D is any connection in TP, the covariant derivative of A (tensor product connection) is referred to (c.f. [Vil]) as the *fundamental form* $\beta(A)$ of A (w.r.t. D and ∇):

$$\beta(A)(Z',Z) = (\phi^{-1}\nabla)_Z(AZ') - A(D_Z Z').$$

<u>Proposition 1</u>

$$T_{\nabla,A}(Z,Z') = \beta(A)(Z',Z)-\beta(A)(Z,Z') + A(T_D(Z,Z'))$$

where T_D is the usual torsion of D. In particular, if D is torsion-free and A has symmetric fundamental form, then $T_{\nabla,A} = 0$.

<u>Proof</u>.

$$T_{\nabla,A}(Z,Z') = (\phi^{-1}\nabla)_Z(AZ') - (\phi^{-1}\nabla)_{Z'}(AZ) - A[Z,Z']$$

$$= \beta(A)(Z',Z) + A(D_Z Z') - \beta(A)(Z,Z') - A(D_{Z'}Z) - A[Z,Z']$$

$$= \beta(A)(Z',Z) - \beta(A)(Z,Z') + A(T_D(Z,Z')). \qquad \square$$

In anticipation of Chapter 3, §1, let $\psi:P \to M$, and consider the product $\psi \times \phi:P \to M \times N$. Define morphisms I_ψ, I_ϕ:

$$\begin{array}{ccc} \psi^{-1}(TM) & \xrightarrow{I_\psi} & \overline{TM} \\ \downarrow & & \downarrow \\ P & \xrightarrow[\psi\times\phi]{} & M\times N \end{array} \quad ; \; I_\psi(p,v) = di_{\phi(p)}\,(\psi(p))(v)$$

for any $p \in P$ and $v \in T_{\psi(p)}M$.

Proposition 2.

(i) $d(\psi \times \phi) = I_\psi \circ d\psi + I_\phi \circ d\phi = (d\psi, d\phi)$

(ii) If (M,g), (N,h) are Riemannian manifolds, then I_ψ and I_ϕ are connection-preserving fibre isometries.

Proof. Let $X \in \mathfrak{C}(TM)$ and $Z \in \mathfrak{C}(TP)$. We observe that:

$$I_\psi \circ \psi^{-1}X = (\psi \times \phi)^{-1}\bar{X}.$$

(i) Firstly, $I_\psi \circ d\psi(Z) + I_\phi \circ d\phi(Z) = \overline{d\psi(Z)} + \overline{d\phi(Z)} = (d\psi(Z), d\phi(Z))$. Now, $\pi_M \circ (\psi \times \phi) = \psi$ and $\pi_N \circ (\psi \times \phi) = \phi$, so that, for any $p \in P$:

$$I_\psi \circ d\psi(Z(p)) + I_\phi \circ d\phi(Z(p)) = di_{\phi(p)} \circ d\pi_M \circ d(\psi \times \phi)(Z(p))$$

$$+ dj_{\psi(p)} \circ d\pi_N \circ d(\psi \times \phi)(Z(p)).$$

For any $(x,y) \in M \times N$, $i_y \circ \pi_M$ projects $M \times N$ onto $M \times \{y\}$, so that $d(i_y \circ \pi_M)(x,y)$ is the projection of $T_{(x,y)}(M \times N)$ onto $\overline{TM}_{(x,y)}$ along $\overline{TN}_{(x,y)}$.

(ii) We show that I_ψ commutes with covariant differentiation on pullback sections:

$$I_\psi(\nabla_Z\psi^{-1}X) = I_\psi(\psi^{-1}\nabla_{d\psi(Z)}X) = (\psi \times \phi)^{-1}(\nabla_{d\psi(Z)}X,0)$$

$$\nabla_Z(I_\psi\circ\psi^{-1}X) = \nabla_Z(\psi\times\phi)^{-1}\bar{X} = (\psi\times\phi)^{-1}\nabla_{(d\psi(Z),d\phi(Z))}\bar{X}, \text{ by (i)}$$

$$= (\psi\times\phi)^{-1}(\nabla_{d\psi(Z)}X,0), \text{ by Proposition 1.1.} \quad \square$$

§3. VARIATIONS

Suppose initially that $M \xrightarrow{\xi} Q \xrightarrow{\zeta} N$, and that E is a vector bundle over N. Then $(\zeta \circ \xi)^{-1}E$ is isomorphic to $\xi^{-1}\zeta^{-1}E$, an isomorphism being:

$$I_{\zeta,\xi}:(q,e) \to (q,(\xi(q),e)), \text{ for } \pi(e) = \zeta \circ \xi(q).$$

Moreover, $(\zeta \circ \xi)^{-1}E$ and $\xi^{-1}\zeta^{-1}E$ are geometrically indistinguisable:

Proposition 1

If E has a connection and fibre metric, then $I_{\zeta,\xi}$ is a connection-preserving isometry.

Proof. We check that $I_{\zeta,\xi}$ commutes with covariant differentiation as pullback sections. If $\alpha \in \mathfrak{C}(E)$ and $W \in \mathfrak{C}(TQ)$:

$$\nabla_W(I_{\zeta,\xi}(\zeta\circ\xi)^{-1}\alpha) = \nabla_W(\xi^{-1}\zeta^{-1}\alpha) = \xi^{-1}(\nabla_{d\xi(W)}(\zeta^{-1}\alpha))$$

$$= \xi^{-1}\zeta^{-1}(\nabla_{d\zeta\circ d\xi(W)}\alpha) = I_{\zeta,\xi}((\zeta\circ\xi)^{-1}\nabla_{d(\zeta\circ\xi)(W)}\alpha)$$

$$= I_{\zeta,\xi}(\nabla_W(\zeta\circ\xi)^{-1}\alpha)$$

where ∇ denotes the connection in E along with its various pullbacks, as appropriate. $\square$

A *variation* of $\phi:M \to N$ is a smooth $\Phi:M \times \mathbb{R} \to N$ with $\Phi(x,0) = \phi(x)$ for all $x \in M$. Then, replacing Q by $M \times \mathbb{R}$ gives the following example of the above:

$$\begin{tikzcd}
\phi_t^{-1}(TN) \arrow[d, "I_{\phi,t}"'] \arrow[drr, "\tilde{\phi}_t"] \arrow[dd, dashed, bend left] & & \\
i_t^{-1}\Phi^{-1}(TN) \arrow[r, "\tilde{i}_t"'] \arrow[d] & \Phi^{-1}(TN) \arrow[r, "\tilde{\Phi}"'] \arrow[d] & TN \arrow[d] \\
M \arrow[r, "i_t"'] & M \times \mathbb{R} \arrow[r, "\Phi"'] & N
\end{tikzcd}$$

By virtue of Proposition 1, we need make no further distinction between $\phi_t^{-1}(TN)$ and $i_t^{-1}\Phi^{-1}(TN)$. If $\frac{\partial}{\partial t}$ denotes the product field $(0,\frac{d}{dt})$ on $M \times \mathbb{R}$, define the *variation field* of Φ by $v = d\Phi \circ \frac{\partial}{\partial t} \in \mathfrak{C}(\Phi^{-1}(TN))$; for each $t \in \mathbb{R}$, put $v_t = i_t^{-1}v \in \mathfrak{C}(\phi_t^{-1}(TN))$. Conversely, any section $v_0 \in \mathfrak{C}(\phi^{-1}(TN))$ determines a variation $\phi_t(x) = \exp t\, v_0(x)$ (subject to a choice of connection in TN) whose variation field agrees with v_0 at $t = 0$.

Concerning variations of tensor fields on M, for each $t \in \mathbb{R}$ let $\hat{i}_t:\mathfrak{T}(M \times \mathbb{R}) \to \mathfrak{T}(M)$ be the extension to $\mathfrak{T}(M \times \mathbb{R})$ of:

$$\hat{i}_t(Z) = Z_t, \text{ for all } Z \in \mathfrak{C}(T(M \times \mathbb{R}))$$

$$\hat{i}_t(\Omega) = i_t^*\Omega, \text{for all } \Omega \in \mathfrak{C}(T^*(M \times \mathbb{R})).$$

For any $B \in \mathfrak{C}(\mathfrak{T}_\ell^k(M \times \mathbb{R}))$, let $B_t = \hat{i}_t B \in \mathfrak{C}(\mathfrak{T}_\ell^k(M))$, and define the *variation field* of B to be $\frac{dB_t}{dt} \in \mathfrak{C}(\mathfrak{T}_\ell^k(M))$, where:

$$\frac{dB_t}{dt}(A) = \frac{d}{dt}(B_t(A)), \text{ for all } A \in \mathfrak{C}(\mathfrak{T}^{\ell}_{k}(M)).$$

Proposition 2

$$\left.\frac{dB_t}{dt}\right|_s = \hat{i}_s \nabla_{\frac{\partial}{\partial t}} B.$$

Proof. We prove the case $k = 2$, $\ell = 0$. If $X,Y \in \mathfrak{C}(TM)$, then:

$$(\nabla_{\frac{\partial}{\partial t}} B)(\bar{X},\bar{Y}) = \nabla_{\frac{\partial}{\partial t}} (B(\bar{X},\bar{Y})) - B(\nabla_{\frac{\partial}{\partial t}} \bar{X},\bar{Y}) - B(\bar{X},\nabla_{\frac{\partial}{\partial t}} \bar{Y})$$

$$= \frac{\partial}{\partial t} B(\bar{X},\bar{Y}) - B(\nabla_{(0,\frac{d}{dt})}(X,0),\bar{Y}) - B(\bar{X},\nabla_{(0,\frac{d}{dt})}(Y,0))$$

$$= \frac{\partial}{\partial t} B(\bar{X},\bar{Y}), \text{ by Proposition 1.1.}$$

Now,

$$B_s(X,Y) = \hat{i}_s B(X,Y) = B(di_s(X), di_s(Y)) = B(\bar{X},\bar{Y})\circ i_s.$$

Thus:

$$(\hat{i}_s \nabla_{\frac{\partial}{\partial t}} B)(X,Y) = (\nabla_{\frac{\partial}{\partial t}} B)(\bar{X},\bar{Y})\circ i_s = \frac{\partial}{\partial t} B(\bar{X},\bar{Y})\circ i_s$$

$$= \lim_{t\to 0} \frac{1}{t} \{B(\bar{X},\bar{Y}) \circ i_{s+t} - B(\bar{X},\bar{Y}) \circ i_s\}$$

$$= \lim_{t\to 0} \frac{1}{t} \{B_{s+t}(X,Y) - B_s(X,Y)\} = \left.\frac{dB_t}{dt}\right|_s (X,Y). \quad \square$$

If h is a Riemannian metric on N, and Φ a variation of $\phi : M \to N$ with variation field v, then Φ^*h is a variation of ϕ^*h with variation field:

Proposition 3

$$\left.\frac{d\phi_t^*h}{dt}\right|_s = 2\ \mathrm{Sym}\ \{h(d\phi_s(E_i),\ \nabla_{E_j} v_s)\ \theta^i \otimes \theta^j\}$$

where $\{E_i\}_1^m$ is any local frame field in M with dual $\{\theta^i\}_1^m$, and Sym denotes "symmetrisation".

Proof. By Proposition 2:

$$\left.\frac{d\phi_t^*h}{dt}\right|_s = \hat{i}_s\ \nabla_{\frac{\partial}{\partial t}}\ \phi^*h = (\hat{i}_s\ \nabla_{\frac{\partial}{\partial t}}\ \phi^*h)(E_i,E_j)\ \theta^i \otimes \theta^j$$

$$(\hat{i}_s\ \nabla_{\frac{\partial}{\partial t}}\ \Phi^*h)(E_i,E_j) = (\nabla_{\frac{\partial}{\partial t}}\Phi^*h)(\bar{E}_i,\bar{E}_j)\circ i_s = \frac{\partial}{\partial t}\cdot\Phi^*h(\bar{E}_i,\bar{E}_j)\circ i_s$$

by Proposition 1.1.

Now, since $(\Phi^{-1}(TN),\ \Phi^{-1}h,\ \Phi^{-1\,N}\nabla)$ is a Riemannian vector bundle (Remark 2.2):

$$\frac{\partial}{\partial t}\cdot\phi^*h(\bar{E}_i,\bar{E}_j) = \frac{\partial}{\partial t}\cdot h(d\phi(\bar{E}_i),d\phi(\bar{E}_j))$$

$$= h(\nabla_{\frac{\partial}{\partial t}}(d\Phi\circ\bar{E}_i),d\Phi(\bar{E}_j)) + h(d\Phi(\bar{E}_i),\ \nabla_{\frac{\partial}{\partial t}}(d\Phi\circ\bar{E}_j)).$$

We note that $d\Phi$ has symmetric fundamental form ([Vil]), ∇ is torsion-free, and $[\bar{E}_i,\ \frac{\partial}{\partial t}] = 0$, so that by Proposition 2.1:

$$\frac{\partial}{\partial t}\cdot\phi^*h(\bar{E}_i,\bar{E}_j) = h(\nabla_{\bar{E}_i}(d\Phi\circ\tfrac{\partial}{\partial t}),d\Phi(\bar{E}_j))+h(d\Phi(\bar{E}_i),\nabla_{\bar{E}_j}(d\Phi\circ\tfrac{\partial}{\partial t}))$$

$$= h(\nabla_{\bar{E}_i} v,\ d\Phi(\bar{E}_j)) + h(d\Phi(\bar{E}_i),\nabla_{\bar{E}_j} v).$$

Now

$$d\Phi(\bar{E}_j) \circ i_s = d\Phi \circ di_s \circ E_j = d\phi_s(E_j)$$

and

$$(\nabla_{\bar{E}_i} v) \circ i_s = i_s^{-1}(\nabla_{\bar{E}_i} v) = \nabla_{E_i}(i_s^{-1} v) = \nabla_{E_i} v_s \ ,$$

so that:

$$\frac{\partial}{\partial t} \cdot \Phi^* h(\bar{E}_i, \bar{E}_j) \circ i_s = h(\nabla_{E_i} v_s, d\phi_s(E_j)) + h(d\phi_s(E_i), \nabla_{E_j} v_s).$$

□

Increasing the number of parameters in a variation gives rise to higher order variation fields. In particular, if $\phi_{s,t}$ is a 2-parameter variation of $\phi : M \to (N,h)$ with variation fields $v = \frac{\partial \Phi}{\partial s}$ and $w = \frac{\partial \Phi}{\partial t}$, then the 2^{nd} order variation fields of $\Phi^* h$ are given by:

Proposition 4.

$$\frac{\partial^2 \phi^*_{s,t} h}{\partial t \ \partial s} = 2 \ \mathrm{Sym} \ \{h(R^N(w, d\phi(E_i))v + \nabla_{E_i} \nabla_{\frac{\partial}{\partial t}} v, d\phi(E_j))$$

$$+ h(\nabla_{E_i} v, \nabla_{E_j} w)\} \ \theta^i \otimes \theta^j.$$

(Omitting subscript (s,t) for notational clarity).

Proof. (cf. [Smi] Proposition 1.1)

$$\left.\frac{\partial^2 \phi^*_{s,t} h}{\partial t \ \partial s}\right|_{(s,t)} = \hat{i}_{s,t} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \Phi^* h, \text{ by Proposition 2}$$

$$= (\hat{i}_{s,t} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \Phi^*h)(E_i,E_j)\theta^i \otimes \theta^j$$

$$= (\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \Phi^*h)(\bar{E}_i,\bar{E}_j) \circ i_{s,t}\, \theta^i \otimes \theta^j$$

$$= \nabla_{\frac{\partial}{\partial t}} [\nabla_{\frac{\partial}{\partial s}} (\Phi^*h(\bar{E}_i,\bar{E}_j))] \circ i_{s,t}\, \theta^i \otimes \theta^j, \text{ by Proposition 1.1.}$$

As in the proof of Proposition 3:

$$\nabla_{\frac{\partial}{\partial s}} (\Phi^*h(\bar{E}_i,\bar{E}_j)) = h(\nabla_{\bar{E}_i} v,\ d\Phi(\bar{E}_j)) + h(d\Phi(\bar{E}_i),\nabla_{\bar{E}_j} v),$$

so that

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} (\Phi^*h(\bar{E}_i,\bar{E}_j)) = h(\nabla_{\frac{\partial}{\partial t}} \nabla_{\bar{E}_i} v, d\Phi(\bar{E}_j)) + h(\nabla_{\bar{E}_i} v, \nabla_{\bar{E}_j} w)$$

$$+ h(\nabla_{\bar{E}_i} w, \nabla_{\bar{E}_j} v) + h(d\Phi(\bar{E}_i), \nabla_{\frac{\partial}{\partial t}} \nabla_{\bar{E}_j} v).$$

Now,

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\bar{E}_i} v - \nabla_{\bar{E}_i} \nabla_{\frac{\partial}{\partial t}} v - \nabla_{[\frac{\partial}{\partial t}, \bar{E}_i]} v = R^N(w, d\Phi(\bar{E}_i))v$$

and

$[\frac{\partial}{\partial t}, \bar{E}_i] = 0$, from which the result follows. □

§4. NEWTON POLYNOMIALS AND NEWTON TENSORS

Consider the following three families $(1 \leq r \leq m)$ in the polynomial ring $\mathbb{R}[x_1,\ldots,x_m]$:

$$\sigma_r(x_1,\dots,x_m) = \sum_{i_1<\dots<i_r} x_{i_1}\dots x_{i_r}$$

$\sigma_r = 0$ if $r > m$

$$\rho_r(x_1,\dots,x_m) = x_1^r + \dots + x_m^r$$

$$f_r(x_1,\dots,x_m) = (-1)^r \sum_{k=1}^{r} (-1)^k \sum_{i_1+\dots+i_k=r} x_{i_1}\dots x_{i_k}$$

(For example,

$$f_1 = x_1, \qquad f_3 = x_1^3 - 2x_1x_2 + x_3$$

$$f_2 = x_1^2 - x_2, \quad f_4 = x_1^4 - 3x_1^2x_2 + 2x_1x_3 + x_2^2 - x_4).$$

<u>Lemma 1.</u> ([Wae] p. 81)

$$\rho_r - \rho_{r-1}\,\sigma_1 + \dots + (-1)^{r-1}\rho_1\sigma_{r-1} + (-1)^r r\sigma_r = 0.$$

□

<u>Lemma 2.</u>

$$x_r - x_{r-1}f_1 + \dots + (-1)^{r-1}x_1f_{r-1} + (-1)^r f_r$$

$$= \sum_{p=0}^{r} (-1)^p\, x_{r-p}\, f_p = 0. \qquad \square$$

Suppose B is a diagonalisable endomorphism of an m-dimensional real vector space.Let $\sigma_r(B)$ and $\rho_r(B)$ denote the corresponding polynomials in the eigenvalues of B, and let $f_r(B) = f_r(\sigma_1(B),\dots,\sigma_m(B))$.

<u>Remark 1.</u> $\rho_t(B)$ = Trace B^r.

In the polynomial ring $\mathbb{R}[x]$, define the r^{th} *Newton polynomial* of B by:

$$\chi_{B,r}(x) = \sum_{p=0}^{r} (-1)^p \sigma_{r-p}(B) x^p = (-1)^r \{x^r - \sigma_1(B) x^{r-1} + \ldots + (-1)^r \sigma_r(B)$$

Remark 2. If $\chi_B(x) = \det(x\mathbb{1} - B)$, the characteristic polynomial of B, then $\chi_{B,m} = (-1)^m \chi_B$.

The Newton polynomials of B satisfy the following recurrence relation:

Lemma 3.

$$\chi_{B,r}(x) = \sigma_r(B) - x\,\chi_{B,r-1}(x). \qquad \square$$

We may use the $f_r(B)$ to "invert" the Newton polynomials:

Proposition 1.

$$x^r = \sum_{=0}^{r} (-1)^p f_{r-p}(B) \chi_{B,p}(x) = (-1)^r \{\chi_{B,r}(x) - f_1(B)\chi_{B,r-1}(x) + \ldots + (-1)^r f_r(B).$$

Proof. By induction on r.
Firstly, $x = -(\sigma_1(B) - x) + \sigma_1(B) = -\chi_{B,1}(x) + f_1(B)$.
Now, by the induction hypothesis:

$$\begin{aligned} x^{r+1} &= (-1)^r \{x\chi_{B,r}(x) - f_1(B) x\chi_{B,r-1}(x) + \ldots + (-1)^r f_r(B)x\} \\ &= (-1)^{r+1} \{\chi_{B,r+1}(x) - f_1(B)\chi_{B,r}(x) + \ldots + (-1)^r f_r(B)\chi_{B,1}(x)\} \\ &+ (-1)^r \{\sigma_{r+1}(B) - f_1(B)\sigma_r(B) + \ldots + (-1)^r f_r(B)\sigma_1(B)\}, \text{ by Lemma 3.} \end{aligned}$$

By Lemma 2, the second summand is just $f_{r+1}(B)$, which completes the induction. $\square$

Since $\chi_{B,r}(B)$ is a polynomial in B, the two endomorphisms have the same eigenspaces. Say the eigenvalues of B are $\mu_1,\ldots,\mu_m \in \mathbb{R}$, and set:

$$\sigma_{r,i}(B) = \sigma_{r,i}(\mu_1,\ldots,\mu_m) = \sigma_r(\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_m)$$

(i.e., delete μ_i).

Lemma 4.

For each i, $\sigma_{r,i}(B) = \chi_{B,r}(\mu_i)$. Thus, $\sigma_{r,i}(B)$ is the i^{th} eigenvalue of $\chi_{B,r}(B)$.

Proof. Induction on r.

$$\sigma_{1,i}(B) = \mu_1 + \ldots + \hat{\mu}_i + \ldots + \mu_m = \sigma_1(B)-\mu_i = \chi_{B,1}(\mu_i)$$

$\sigma_{r+1,i}(B) = \sigma_{r+1}(B)-\mu_i\sigma_{r,i}(B)$, by the definition of

$\sigma_{r,i}(B)$, $\sigma_{r+1,i}(B)$

$= \sigma_{r+1}(B) - \mu_i\, \chi_{B,r}(\mu_i)$, by the induction hypothesis

$= \chi_{B,r+1}(\mu_i)$, by Lemma 3. □

From Lemma 4, we have that $\sigma_{r,i}(B) = \sum_{p=0}^{r} (-1)^p \sigma_{r-p}(B)\mu_i^p$.

We now define

$$\sigma_{r,i,j}(B) = \sum_{p=0}^{r} (-1)^p\, \sigma_{r-p,i}(B)\mu_j^p$$

and so on for longer strings of suffices.

Proposition 2.

(i) $\sigma_{r,i,j}(B) = \sigma_{r,i}(B) - \mu_j\, \sigma_{r-1,i,j}(B)$.

(ii) $\sigma_{r,i,j}(B) = \sigma_{r,j,i}(B)$.

(iii) If $i \neq j$, then $\sigma_{r,i,j}(B) = \sigma_r(\mu_1,\ldots,\hat{\mu}_i,\ldots,\hat{\mu}_j,\ldots,\mu_m)$.

Proof.

(ii) We have that $\sigma_{r,i,j}(B) = \sum\limits_{p=0}^{r} (-1)^p \sigma_{r-p}(B) \sum\limits_{s+t=p} \mu_i^s\, \mu_j^t$

which is symmetric in i,j.

(iii) Induction on r.

$$\sigma_{1,i,j}(B) = \sigma_1(B)-\mu_i-\mu_j = \sigma_1(\mu_1,\ldots,\hat{\mu}_i,\ldots,\hat{\mu}_j,\ldots,w_m)$$

$$\sigma_{r+1,i,j}(B) = \sigma_{r+1,i}(B) - \mu_j\, \sigma_{r,i,j}(B), \text{ by (i)}$$

$$= \sigma_{r+1}(\mu_1,\ldots,\hat{\mu}_i,\ldots,\hat{\mu}_j,\ldots,\mu_m), \text{ by the induction hypothesis.} \quad \square$$

We shall abbreviate $\chi_{B,r}(B)$ to $\chi_r(B)$, the r^{th} *Newton tensor of* B.

Proposition 3.

Trace $\chi_r(B) = (m-r)\, \sigma_r(B)$.

Proof.

$$\text{Trace } \chi_r(B) = (-1)^r \text{ Trace } \{B^r - \sigma_1(B)B^{r-1} + \ldots + (-1)^r \sigma_r(B)\}$$

$= (-1)^r\{\rho_r(B)-\sigma_1(B)\rho_{r-1}(B)+\ldots+(-1)^r m\sigma_r(B)\}$, by Remark 1

$= (-1)^r\{(-1)^r(m-r)\sigma_r(B)\}$, by Lemma 1

$= (m-r)\sigma_r(B)$. □

Let $\{B_t\}$ be a 1-parameter family of diagonalisable endomorphisms. The variations produced in the associated elementary symmetric polynomials and Newton tensors have the following variation fields:

Lemma 5. ([Rei] Lemma A)

$$\frac{d}{dt}\Big|_s \sigma_{r+1}(B_t) = \mathrm{Trace}\,[\frac{dB_t}{dt}\Big|_s \circ \chi_r(B_s)]. \quad □$$

Lemma 6.

$$\frac{d}{dt}\chi_{r+1}(B_t) = \sum_{p+q=r} (-1)^q\{\frac{d}{dt}\sigma_{p+1}(B_t)B^q - B^q\frac{dB_t}{dt}\chi_p(B)\}.$$

Proof. An easy induction using Lemma 3. □

Applying Proposition 1 to Lemma 6:

Lemma 7.

$$\frac{d}{dt}\chi_{r+1}(B_r) = \sum_{p+q+s=r} (-1)^s f_s(B)\{\frac{d}{dt}\sigma_{p+1}(B_t)\chi_q(B)$$

$$- \chi_q(B)\frac{dB_t}{dt}\chi_p(B)\}. \quad □$$

§5. HIGHER-POWER ENERGY

Let V be an m-dimensional real vector space with inner product g, and B a bilinear form on V. Identifying $V^* \otimes V^*$ with Hom (V,V^*) and noting that the non-degeneracy of g provides an inverse $g^{-1} \in$ Hom (V^*,V), we construct the endomorphism $g^{-1}B$ of V.

Lemma 1.

$$g(g^{-1}B(v),w) = B(v,w), \text{ for all } v,w \in V. \quad \square$$

If now (M,g) is a compact, oriented, boundaryless, m-dimensional Riemannian manifold and $\phi:(M,g) \to (N,h)$, we may form the (1,1) tensor $g^{-1}\phi^*h$.

Proposition 1.

Trace $g^{-1}\phi^*h = 2e(\phi)$, where $e(\phi) = \frac{1}{2}\,||d\phi||^2$ is the *energy density* of ϕ.

Proof. If $\{E_i\}_1^m$ is a g-orthonormal frame:

$$\text{Trace } g^{-1}\phi^*h = \sum_i g(g^{-1}\phi^*h(E_i),E_i) = \sum_i \phi^*h(E_i,E_i), \text{ by Lemma 1,}$$

$$= \sum_i h(d\phi(E_i),d\phi(E_i)) = ||d\phi||^2. \quad \square$$

Accordingly, we apply the algebra of §4 to each tangent space of M and define the r^{th} *energy density* of ϕ by:

$$\sigma_r(\phi) = \sigma_r(\phi,g) = \sigma_r(g^{-1}\phi^*h)$$

The r^{th} *energy of* ϕ is then:

$$E_r(\phi) = E_r(\phi,g) = \int_M \sigma_r(\phi,g)v_g$$

where v_g is the volume element of (M,g).

Remark 1. If $\{E_i\}$ is a g-orthonormal frame which is ϕ^*h-orthogonal, then:

$$g^{-1}\phi^*h(E_i) = \sum_j g(g^{-1}\phi^*h(E_i),E_j)E_j = \sum_j \phi^*h(E_i,E_j)E_j,$$

by Lemma 1

$$= \phi^*h(E_i,E_i)E_i = h(d\phi(E_i),d\phi(E_i))E_i.$$

Thus, $\{E_i\}$ is a frame of eigenvectors of $g^{-1}\phi^*h$, with corresponding eigenvalues $\{ \|d\phi(E_i)\|^2 \}$. In particular, the the higher-power energy densities are non-negative.

Proposition 2

$$\sigma_r(\phi)(x) = 0 \text{ iff rank } d\phi(x) < r, \text{ for all } x \in M.$$

Thus, $E_r(\phi) = 0$ iff rank $\phi < r$.

Proof. Since all the eigenvalues of $g^{-1}\phi^*h$ are non-negative:

$\sigma_r(\phi)(x) = 0$ iff $g^{-1}\phi^*h(x)$ has at most $r-1$ non-zero eigenvalues
iff rank $d\phi(x) < r$, by Remark 1. □

It is well known that when $n = 2$ the energy functional is conformally invariant i.e. invariant under conformal changes of metric on M ([E-S] p. 126). This property generalises to the higher-power energies, where it is seen to be dependent on the following transformation law for volume elements:

Lemma 2.

If g and g' are Riemannian metrics on M, then:

$$v_{g'} = \det (g^{-1}g')^{\frac{1}{2}} v_g.$$

Note. The $(1,1)$ tensor $g^{-1}g'$ has positive eigenvalues (cf. Remark 1), and therefore admits a square root.

Proof. The volume element of g' is characterised as that unique m-form $v_{g'}$ satisfying $v_{g'}(E_1',\ldots,E_m') = 1$, for any positively oriented g'-orthonormal frame $\{E_i'\}$. By Lemma 1:

$$g((g^{-1}g')^{\frac{1}{2}}E_i',(g^{-1}g')^{\frac{1}{2}}E_j') = g(g^{-1}(g')^{\frac{1}{2}}E_i',g^{-1}(g')^{\frac{1}{2}}E_j')$$

$$= g(g^{-1}g'E_i',E_j') = g'(E_i',E_j') = \delta_{ij}$$

so that $\{(g^{-1}g')^{\frac{1}{2}}E_i'\}$ is a positively oriented g-orthonormal frame. The result now follows from the transformation law ([Spi] vol. 1, Ch. 7, Thm. 5):

$$\omega(Pv_1,\ldots,Pv_m) = \det P.\omega(v_1,\ldots,v_m)$$

for m-forms ω and $(1,1)$ tensors P. □

Proposition 3

$E_r(\phi)$ is conformally invariant iff $m = 2r$, or rank $\phi < r$.

Proof. By Proposition 2, rank $\phi < r$ iff $E_r(\phi,g) = 0$ for any metric g, in which case $E_r(\phi)$ is certainly conformally invariant.

Also, if $m = 2r$ and $\lambda : M \to \mathbb{R}$ is smooth and strictly positive:

$$E_r(\phi,\lambda g) = \int_M \sigma_r(\phi,\lambda g)v_{\lambda g} = \int_M \sigma_r(\phi,\lambda g)\lambda^r v_g, \text{ by Lemma 2}$$

$$= \int_M \frac{1}{\lambda^r}\sigma_r(\phi,g)\lambda^r v_g = E_r(\phi,g).$$

Conversely, if $E_r(\phi)$ is conformally invariant, then, for any strictly positive smooth $\lambda: M \to \mathbb{R}$, it follows from Lemma 2 that:

$$\int_M \frac{1}{\lambda^r}\sigma_r(\phi,g)\lambda^{m/2}v_g = E_r(\phi,\lambda g) = E_r(\phi,g) = \int_M \sigma_r(\phi,g)v_g$$

Thus,

$$\int_M \sigma_r(\phi,g)(\lambda^{m/2-r}-1)v_g = 0. \qquad \lambda^{\frac{M}{2}-r}$$

Assume that $E_r(\phi) \neq 0$, so that $\sigma_r(\phi)$ is somewhere positive (and nowhere negative). If $m > 2r$ (resp. $m < 2r$), then choosing $\lambda < 1$ (resp. $\lambda > 1$) gives a contradiction, so that $m = 2r$. □

Define the *volume density* of $\phi:(M,g) \to (N,h)$ by:

$$\text{vol}\ (\phi) = \text{vol}\ (\phi,g) = \sigma_m(\phi,g)^{\frac{1}{2}}$$

and the *volume* of ϕ by:

$$\text{Vol}\ (\phi) = \text{Vol}\ (\phi,g) = \int_M \text{vol}\ (\phi)v_g.$$

Remark 2. vol (ϕ) is the factor through which ϕ alters infinitesimal m-dimensional volume. For, if $\{E_i\}$ is a g-orthonormal frame (hence spanning a parallelopiped of unit volume in (M,g)) which is also ϕ^*h-orthogonal, then by Remark 1:

Volume of parallelopiped spanned by

$$\{d\phi(E_1),\dots,d\phi(E_m)\} = \|d\phi(E_1)\|\cdots\|d\phi(E_m)\|$$

$$= \det\,(g^{-1}\phi^*h)^{\frac{1}{2}} = \sigma_m(\phi,g)^{\frac{1}{2}}.$$

In particular, Vol $(\phi) = 0$ unless ϕ at some point attains maximal rank.

It is well known that when $m = 2$ the energy density majorises the volume density, equality coinciding with weak conformality ([E-S] p. 126; [Lem]). To generalise this say that $\phi:(M,g) \to (N,h)$ is r-*conformal* if ϕ is conformal on an open subset (possibly empty), away from which rank $\phi < r$.

<u>Remark 3</u>. 1-conformality is equivalent to weak conformality.

<u>Proposition 4</u>.

Suppose that $m = 2r$. Then, $\sigma_r(\phi) \geqslant \binom{m}{r}$ vol (ϕ), with equality precisely when ϕ is r-conformal.

<u>Proof</u>. Let $\{E_i\}$ be a g-orthonormal frame diagonalising $g^{-1}\phi^*h$, and put $\lambda_i = \|d\phi(E_i)\|$. By Remark 1:

$$0 \leqslant \sum_{\mu\in S_m} (\lambda_{\mu(1)}\cdots\lambda_{\mu(r)} - \lambda_{\mu(r+1)}\cdots\lambda_{\mu(m)})^2$$

$$= (r!)^2\{\sigma_r(\phi) - \tbinom{m}{r}\lambda_1\cdots\lambda_m\} = (r!)^2\{\sigma_r(\phi) - \tbinom{m}{r}\text{ vol }(\phi)\}.$$

If ϕ is r-conformal, then either $\phi^*h(x) = \lambda(x)g(x)$ (so that $\lambda_1(x) = \dots = \lambda_m(x) = \lambda(x)$), or rank $d\phi(x) < r$ (so that at least $r+1$ of the $\lambda_i(x)$ vanish). In either case,

$\lambda_{\mu(1)} \cdots \lambda_{\mu(r)} - \lambda_{\mu(r+1)} \cdots \lambda_{\mu(m)} = 0$ for all $\mu \in S_m$, so that $\sigma_r(\phi) = \binom{m}{r}$ vol (ϕ).

Conversely, if $\sigma_r(\phi) = \binom{m}{r}$ vol (ϕ), we have the system of equations:

$$\lambda_{\mu(1)} \cdots \lambda_{\mu(r)} - \lambda_{\mu(r+1)} \cdots \lambda_{\mu(m)} = 0, \text{ for all } \mu \in S_m.$$

If one of the $\lambda_i(x)$ vanishes, then so do at least r others. On the other hand, if no $\lambda_i(x)$ vanishes, they are all equal. Thus, ϕ is r-conformal. □

§6. THE FIRST VARIATION OF $E_r(\phi_t, g)$ - TENSION FIELDS

Lemma 1. ("Integration by Parts")

Let $(E, \nabla, \langle,\rangle)$ be a Riemannian vector bundle over (M,g) (compact). If $\alpha \in \mathcal{C}(E)$ and $\beta \in \mathcal{C}(T^*M \otimes E)$, then:

$$\int_M \langle \nabla\alpha, \beta \rangle v_g = - \int_M \langle \alpha, \text{Trace } \nabla\beta \rangle v_g.$$

Proof. The exterior differential (d) and codifferential (d*) on E-valued differential forms act on 0-forms α, and 1-forms β, as follows ([E-L] Ch. 1):

$$d\alpha = \nabla\alpha, \quad d^*\beta = - \text{Trace } \nabla\beta.$$

But, d* is characterised as the L^2-adjoint of d. □

Applying the algebra of §4 to each tangent space of M, define the r^{th} *Newton tensor* of $\phi: (M,g) \to (N,h)$ by:

$$\chi_r(\phi) = \chi_r(\phi,g) = \chi_r(g^{-1}\phi^*h).$$

Now, let ϕ_t be a variation of ϕ, with variation field v. If <,> denotes the tensor product metric on $T^*M \otimes \phi^{-1}(TN)$:

<u>Lemma 2.</u>

$$\frac{d}{dt}\Big|_s \sigma_r(\phi_t) = 2 \langle d\phi_s \circ \chi_{r-1}(\phi_s), \nabla v_s\rangle.$$

<u>Proof</u>. Let $\{E_i\}$ be a g-orthonormal frame. By Proposition 3.3 and Lemma 4.5:

$$\frac{d}{dt}\Big|_s \sigma_r(\phi_t) = \text{Trace } [g^{-1} \frac{d\phi_t^*h}{dt}\Big|_s \chi_{r-1}(\phi_s)]$$

$$= \sum_{i,j} (\frac{d\phi_t^*h}{dt}\Big|_s)_{ji} \chi_{r-1}(\phi_s)^j_i$$

$$= \sum_{i,j} \{h(d\phi_s(E_j),\nabla_{E_i}v_s) + h(d\phi_s(E_i),\nabla_{E_j}v_s)\}\chi_{r-1}(\phi_s)^j_i.$$

If in addition $\{E_i\}$ is chosen ϕ_s^*h-orthogonal, then $\{E_i\}$ diagonalises both $g^{-1}\phi_s^*h$ and $\chi_{r-1}(\phi_s)$, so that

$$\frac{d}{dt}\Big|_s \sigma_r(\phi_t) = 2 \sum_i h(d\phi_s(E_i),\nabla_{E_i}v_s)\chi_{r-1}(\phi_s)^i_i$$

$$= 2 \sum_i h(d\phi_s(\chi_{r-1}(\phi_s)^i_i E_i),\nabla_{E_i}v_s) = 2 \sum_i h(d\phi_s \circ \chi_{r-1}(\phi_s)E_i,\nabla_{E_i}v_s)$$

$$= 2 \langle d\phi_s \circ \chi_{r-1}(\phi_s), \nabla v_s\rangle. \qquad \square$$

<u>Proposition 1.</u>

$$\frac{d}{dt}\Big|_s E_r(\phi_t) = -2 \int_M h(\text{Trace } \nabla(d\phi_s \circ \chi_{r-1}(\phi_s)),v_s)v_g.$$

Proof.

$$\frac{d}{dt}\Big|_s E_r(\phi_t) = \int_M \frac{d}{dt}\Big|_s \sigma_r(\phi_t)v_g = 2\int_M \langle d\phi_s \circ \chi_{r-1}(\phi_s), \nabla v_s\rangle v_g,$$

by Lemma 2

$$= -2\int_M h\ (\mathrm{Trace}\ \nabla(d\phi_s \circ \chi_{r-1}(\phi_s)), v_s)v_g,$$

by Lemma 1. □

Define the r^{th} *tension field* of ϕ (w.r.t.g) by:

$$\tau_r(\phi) = \tau_r(\phi,g) = \mathrm{Trace}\ \ \nabla(d\phi \circ \chi_{r-1}(\phi)).$$

Theorem 1.

ϕ is a critical point of E_r w.r.t.g iff $\tau_r(\phi,g) = 0$. □

Say that ϕ is *r-harmonic* (w.r.t.g) whenever $\tau_r(\phi,g) = 0$.

Remark. 1-harmonicity is equivalent to harmonicity; for $\chi_0(\phi,g) = 1_{TM}$ so that $\tau_1(\phi) = \tau(\phi)$ (the usual tension field of ϕ).

By Proposition 5.2, if rank $\phi < r$ then (ϕ,g) is a minimum for E_r. Thus:

Corollary 1.

If rank $\phi < r$, then ϕ is r-harmonic w.r.t. any metric on M. □

With respect to coordinates (x^i) in M and (y^α) in N, we have (cf. [E-L] p. 27):

$$\tau_{r+1}(\phi,g)^\gamma = g^{ij}\ \{\frac{\partial}{\partial x^i}(\phi_k^\gamma\ \chi_r(\phi)_j^k) - \phi_\ell^\gamma\ \chi_r(\phi)_k^\ell\ {}^M\Gamma_{ij}^k$$
$$+ \phi_i^\alpha\ \phi_\ell^\beta \chi_r(\phi)_j^\ell\ {}^N\Gamma_{\alpha\beta}^\gamma\}.$$

The vanishing of $\tau_{r+1}(\phi,g)$ thus determines a symmetric, second order, quasi-linear system of n equations (n = dim N), which we may write in the form:

$$\tau_{r+1}(\phi)^{\gamma} = P_{\gamma\beta}(\phi,\partial)\phi^{\beta} + F_{\gamma}(\phi) = 0.$$

Here, each *principal part* $P_{\gamma\beta}(\phi,\partial)$ is a homogeneous polynomial of degree two in the partial differentials $\partial_1,\ldots,\partial_m$, whose coefficients, along with the $F_{\gamma}(\phi)$, are functions of the ϕ^{α} and their first derivatives. The differential operator τ_{r+1} is said to be *elliptic* on some submanifold $\mathcal{S} \subset C^2(M,N)$ if the associated quadratic forms $P_{\gamma\beta}(\phi,v)$ on $\mathbb{R}^n$ are non-degenerate, for all $v \in \mathbb{R}^m \setminus \{0\}$ and $\phi \in \mathcal{S}$. (cf. [Hor] p. 268). It is well known that τ_1 is elliptic on the whole of $C^2(M,N)$.

Henceforward, suppose that (x^i) has been chosen such that $\{\frac{\partial}{\partial x^i}\Big|_x\}_1^m$ is a g-orthonormal frame which is also ϕ^*h-orthogonal, for some $x \in M$. Write $\lambda_i = \|d\phi(\frac{\partial}{\partial x^i}\Big|_x)\|$ and re-order the x^i so that $\lambda_1 > \ldots > \lambda_m$; then λ_i^2 is the ith eigenvalue of $g^{-1}\phi^*h(x)$ (Remark 5.1). If rank $d\phi(x) = \rho$, define $F_i = \frac{1}{\lambda_i} d\phi(\frac{\partial}{\partial x^i}\Big|_x)$ for $1 < i < \rho$, and extend to an h-orthonormal frame $\{F_{\alpha}\}_1^n$ at $\phi(x)$. Now let (y^{α}) be the normal coordinate chart determined by $\{F_{\alpha}\}$.

<u>Lemma 3</u>.

$$\frac{\partial}{\partial x^i}\Big|_x \sigma_r(\phi) = \sum_p \sigma_{r-1,p}(\phi) \frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi^*h)_p^p.$$

<u>Proof</u>. Write $g^{-1}\phi^*h = B$. Then

$$\sigma_r(\phi) = \sum_{p_1<\cdots<p_r} \varepsilon^{q_1\cdots q_r}_{p_1\cdots p_r} B^{p_1}_{q_1} \cdots B^{p_r}_{q_r}$$

where the *generalised Kronecker symbols* are defined:

$$\varepsilon^{q_1\cdots q_r}_{p_1\cdots p_r} = \operatorname{sgn}\begin{pmatrix} q_1\cdots q_r \\ p_1\cdots p_r \end{pmatrix}, \text{ if the } p_i \text{ are distinct and the } q_i \text{ permute the } p_i$$

$$= 0, \text{ otherwise.}$$

$$\frac{\partial}{\partial x^i}\sigma_r(\phi) = \sum_{p_1<\cdots<p_r} \varepsilon^{q_1\cdots q_r}_{p_1\cdots p_r}\left\{\frac{\partial B^{p_1}_{q_1}}{\partial x^i} B^{p_2}_{q_2}\cdots B^{p_r}_{q_r} + \cdots + B^{p_1}_{q_1}\cdots B^{p_{r-1}}_{q_{r-1}}\frac{\partial B^{p_r}_{q_r}}{\partial x^i}\right\}.$$

In our chosen coordinates, evaluating at x forces $\begin{pmatrix} q_1\cdots q_r \\ p_1\cdots p_r \end{pmatrix}$ = identity. Thus:

$$\left.\frac{\partial}{\partial x^i}\right|_x \sigma_r(\phi) = \sum_{p_1<\cdots<p_r}\left\{\left.\frac{\partial B^{p_1}_{p_1}}{\partial x^i}\right|_x \lambda^2_{p_2}\cdots\lambda^2_{p_r} + \cdots + \lambda^2_{p_1}\cdots\lambda^2_{p_r}\left.\frac{\partial B^{p_r}_{p_r}}{\partial x^i}\right|_x\right\}$$

$$= \sum_p \sigma_{r-1,p}(\phi)\left.\frac{\partial}{\partial x^i}\right|_x (g^{-1}\phi^*h)^p_p. \quad \square$$

<u>Lemma 4</u>.

$$\left.\frac{\partial}{\partial x^i}\right|_x \chi_r(\phi)^k_j = \sum_p \sigma_{r-1,j,p}(\phi)\left.\frac{\partial}{\partial x^i}\right|_x (g^{-1}\phi^*h)^p_p\delta^k_j$$

$$- \sigma_{r-1,j,k}(\phi)\left.\frac{\partial}{\partial x^i}\right|_x (g^{-1}\phi^*h)^k_j.$$

<u>Proof</u>. Induction on r.

Firstly,

$$\frac{\partial}{\partial x^i}\Big|_x \chi_1(\phi)^k_j = \frac{\partial}{\partial x^i}\Big|_x \{\sigma_1(\phi)\delta^k_j - (g^{-1}\phi{*}h)^k_j\}$$

$$= \frac{\partial}{\partial x^i}\Big|_x \{\text{Trace } (g^{-1}\phi{*}h)\delta^k_j - (g^{-1}\phi{*}h)^k_j\}$$

$$= \sum_p \frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^p_p\ \delta^k_j - \frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^k_j.$$

Now, by Lemma 4.3, $\chi_{r+1}(\phi)^k_j = \sigma_{r+1}(\phi)\delta^k_j - (g^{-1}\phi{*}h)^k_\ell\ \chi_r(\phi)^\ell_j$.

Thus:

$$\frac{\partial}{\partial x^i}\Big|_x \chi_{r+1}(\phi)^k_j = \frac{\partial}{\partial x^i}\Big|_x \sigma_{r+1}(\phi)\delta^k_j - \sigma_{r,j}(\phi)\ \frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^k_j$$

$$- \lambda^2_k \frac{\partial}{\partial x^i}\Big|_x \chi_r(\phi)^k_j$$

$$= \sum_p \sigma_{r,p}(\phi)\ \frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^p_p\ \delta^k_j - \sigma_{r,j}(\phi)\frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^k_j$$

$$- \sum_p \lambda^2_j \sigma_{r-1,j,p}(\phi)\ \frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^p_p\ \delta^k_j + \lambda^2_k\ \sigma_{r-1,j,k}(\phi)\frac{\partial}{\partial x^i}\Big|_x (g^{-1}\phi{*}h)^k_j$$

by Lemma 3 and the induction hypothesis. Plugging-in the recurrence relation of Proposition 4.2 completes the induction. □

The matrix of principal parts of $\tau_{r+1}(\phi)$ is now given by:

<u>Proposition 2</u>.

$$P_{\gamma\beta}(\phi,\partial)(x) = \sum_i \{\sigma_{r,i}(\phi)\delta_{\gamma\beta} - \sum_{k\neq i} \sigma_{r-1,i,k}(\phi)\phi^\gamma_k\ \phi^\beta_k\}\ \partial_i\partial_i$$

$$+ \sum_{i<j} \sigma_{r-1,i,j}(\phi)(\phi^\gamma_i\ \phi^\beta_j + \phi^\gamma_j\ \phi^\beta_i)\partial_i\partial_j.$$

Proof. By Lemma 4:

$$\frac{\partial}{\partial x^i}\bigg|_x \phi^\gamma_k \, \chi_r(\phi)^k_j = \frac{\partial^2\phi^\gamma}{\partial x^i \partial x^j}\,\sigma_{r,j}(\phi)$$

$$+ \sum_k \sigma_{r-1,j,k}(\phi)\ \{\phi^\gamma_j \frac{\partial}{\partial x^i}\bigg|_x (g^{-1}\phi^*h)^k_k - \phi^\gamma_k \frac{\partial}{\partial x^i}\bigg|_x (g^{-1}\phi^*h)^k_j\}.$$

Now, $(g^{-1}\phi^*h)^k_j = h_{\alpha\beta}\,\phi^\alpha_j\,\phi^\beta_\ell\, g^{\ell k}$, so that:

$$\frac{\partial}{\partial x^i}\bigg|_x (g^{-1}\phi^*h)^k_j = \sum_\alpha \left(\phi^\alpha_j \frac{\partial^2\phi^\alpha}{\partial x^i \partial x^k} + \phi^\alpha_k \frac{\partial^2\phi^\alpha}{\partial x^i \partial x^j}\right) + \text{lower order terms.}$$

Thus,

$$\frac{\partial}{\partial x^i}\bigg|_x \phi^\gamma_k \, \chi_r(\phi)^k_j = \frac{\partial^2\phi^\gamma}{\partial x^i \partial x^j}\,\sigma_{r,j}(\phi)$$

$$+ \sum_{\alpha,k} \sigma_{r-1,j,k}(\phi)\{(2\phi^\gamma_j\phi^\alpha_k - \phi^\gamma_k\phi^\alpha_j)\frac{\partial^2\phi^\alpha}{\partial x^i \partial x^k} - \phi^\gamma_k\,\phi^\alpha_k \frac{\partial^2\phi^\alpha}{\partial x^i \partial x^j}\}$$

$$+ \text{ lower order terms.}$$

The leading term of $\tau_{r+1}(\phi)^\gamma = 0$ is

$$\sum_i \frac{\partial}{\partial x^i}\bigg|_x (\phi^\gamma_k\,\chi_r(\phi)^k_i), \quad \text{from which:}$$

$$P_{\gamma\gamma}(\phi,\partial)(x) = \sum_i \{\sigma_{r,i}(\phi) - \sum_k \sigma_{r-1,i,k}(\phi)(\phi^\gamma_k)^2 + \sigma_{r-1,i,i}(\phi)(\phi^\gamma_i)^2\}\partial_i\partial_j$$

$$+ \sum_{i<j} 2\,\sigma_{r-1,i,j}(\phi)\phi^\gamma_i\,\phi^\gamma_j\,\partial_i\partial_j$$

$$= \sum_i \{\sigma_{r,i}(\phi) - \sum_{k\neq i} \sigma_{r-1,i,k}(\phi)(\phi^\gamma_k)^2\}\partial_i\partial_i$$

$$+ 2 \sum_{i<j} \sigma_{r-1,i,j}(\phi)\phi_i^\gamma \phi_j^\gamma \partial_i\partial_j .$$

$$P_{\gamma\beta}(\phi,\partial)(x) = \sum_i \{\sigma_{r-1,i,i}(\phi)\phi_i^\gamma\phi_i^\beta - \sum_k \sigma_{r-1,i,k}(\phi)\phi_k^\gamma \phi_k^\beta\} \partial_i\partial_i$$

$$+ \sum_{i<j} \sigma_{r-1,i,j}(\phi)\{2\phi_i^\gamma\phi_j^\beta - \phi_j^\gamma\phi_i^\beta + 2\phi_j^\gamma\phi_i^\beta - \phi_i^\gamma\phi_j^\beta\} \partial_i\partial_j$$

$$= - \sum_i \sum_{k\neq i} \sigma_{r-1,i,k}(\phi)\phi_k^\gamma \phi_k^\beta \partial_i\partial_i$$

$$+ \sum_{i<j} \sigma_{r-1,i,j}(\phi)(\phi_i^\gamma \phi_j^\beta + \phi_j^\gamma \phi_i^\beta)\partial_i\partial_j . \quad \square$$

Corollary 2.

(a) When either of $\gamma, \beta > \rho$:-

$$P_{\gamma\beta}(\phi,\partial)(x) = 0 \quad (\gamma \neq \beta)$$

$$P_{\gamma\gamma}(\phi,\partial)(x) = \sum_i \sigma_{r,i}(\phi)\partial_i\partial_i$$

(b) When both $\gamma, \beta < \rho$:-

$$P_{\gamma\beta}(\phi,\partial)(x) = \sigma_{r-1,\gamma,\beta}(\phi)\lambda_\gamma\lambda_\beta\partial_\gamma\partial_\beta \quad (\gamma \neq \beta)$$

$$P_{\gamma\gamma}(\phi,\partial)(x) = \sigma_{r-1,\gamma,\gamma}(\phi) \lambda_\gamma^2\partial_\gamma\partial_\gamma + \sum_i \sigma_{r,i,\gamma}(\phi)\partial_i\partial_i$$

Proof.

(b) $$P_{\gamma\gamma}(\phi,\partial)(x) = \sum_{i\neq\gamma} (\sigma_{r+i}(\phi)-\lambda_\gamma^2\sigma_{r-1,i,\gamma}(\phi))\partial_i\partial_i + \sigma_{r,\gamma}(\phi)\partial_\gamma\partial_\gamma$$

$$= \sum_{i\neq\gamma} \sigma_{r,i,\gamma}(\phi)\partial_i\partial_i + \sigma_{r,\gamma}(\phi)\partial_\gamma\partial_\gamma ,$$ by Proposition 4.2(i)

$$= \sum_i \sigma_{r,i,\gamma}(\phi)\partial_i\partial_i + \sigma_{r-1,\gamma,\gamma}(\phi)\lambda_\gamma^2\partial_\gamma\partial_\gamma ,$$ by Proposition 4.2(i). $\square$

Now, for any $v \in \mathbb{R}^m$ the quadratic form $P_{\gamma\beta}(\phi,v)(x)$ has matrix:

$$\begin{array}{cc} \left[\begin{array}{c:c} A(\phi,v) = A'(\phi,v) + A''(\phi,v) & \mathbb{O} \\ \hdashline \mathbb{O} & D(\phi,v) \end{array}\right] & \begin{array}{c} \rho \\ \cdots\cdots \\ n-\rho \end{array} \\ \begin{array}{cc} \qquad\qquad \rho \qquad\qquad & \qquad n-\rho \end{array} & \end{array}$$

where

$$A'(\phi,v) = \operatorname{diag} \{\sum_i \sigma_{r,i,1}(\phi)v_i^2,\ldots, \sum_i \sigma_{r,i,\rho}(\phi)v_i^2\}$$

$$A''(\phi,v) = (\sigma_{r-1,i,j}(\phi)\ \lambda_i\ \lambda_j\ v_iv_j)_{1<i,\ j<\rho}$$

$$D(\phi,v) = \operatorname{diag} \{\sum_i \sigma_{r,i}(\phi)v_i^2,\ldots, \sum_i \sigma_{r,i}(\phi)v_i^2\}.$$

Clearly, $P_{\gamma\beta}(\phi,v)(x)$ is non-degenerate iff $A(\phi,v)$ is non-degenerat

iff $\det A(\phi,v) \neq 0$.

If $\rho < r+1$, then $A(\phi,v) = 0$. Moreover, when $\rho = r+1$ we have:

$$A'(\phi,v) = \operatorname{diag} \{\lambda_2^2 \cdots\lambda_{r+1}^2 \sum_{i>r+1} v_i^2,\ldots,\lambda_1^2\cdots\lambda_r^2 \sum_{i>r+1} v_i^2\}$$

$$A''(\phi,v) = (\lambda_1^2\cdots\lambda_{i-1}^2\lambda_i\lambda_{i+1}^2\cdots\lambda_{j-1}^2\lambda_j\lambda_{j+1}^2\cdots\lambda_{r+1}^2 v_iv_j)_{1<i,\ j<\rho}$$

$$\det A(\phi,v) = \lambda_1^{2r}\cdots\lambda_{r+1}^{2r} \begin{vmatrix} v_1^2 + \sum_{i>r+1} v_i^2 & v_1v_2 & \cdots\cdots & v_1v_{r+1} \\ v_2v_1 & v_2^2 + \sum_{i>r+1} v_i^2 & \cdots & v_2v_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r+1}v_1 & v_{r+1}v_2 & \cdots & v_{r+1}^2+\sum_{i>r+1} v_i^2 \end{vmatrix}$$

so that, for example, $\det A(\phi,(v_1,\ldots,v_{r+1},0,\ldots,0)) = 0$.

When r=2, we observe that the diagonal elements of $A(\phi,v)$ may be re-written:

$$A(\phi,v)_{\gamma\gamma} = \sum_{i\neq\gamma} \sigma_{1,i,\gamma}(\phi)v_i^2 + \sigma_{1,\gamma}(\phi)v_\gamma^2 \quad (\gamma \leq \rho)$$

$$= \sum_i (\lambda_1^2 + \ldots + \hat{\lambda}_1^2 + \ldots + \hat{\lambda}_\gamma^2 + \ldots + \lambda_m^2)v_i^2$$

$$= \sum_{i\neq\gamma} \lambda_i^2(v_1^2 +\ldots+\hat{v}_i^2 +\ldots+ v_m^2) = \sum_{i\neq\gamma} \lambda_i^2 \|\hat{v}_i\|^2$$

where $\hat{v}_i \in \mathbb{R}^m$ is the vector $(v_1,\ldots,v_{i-1},0,v_{i+1},\ldots,v_m)$. Thus:

$$A(\phi,v) = \begin{bmatrix} \sum_{i\neq 1} \lambda_i^2 \|\hat{v}_i\|^2 & \lambda_1\lambda_2 v_1 v_2 & \cdots & \lambda_1\lambda_\rho v_1 v_\rho \\ & \ddots & & \vdots \\ \lambda_1\lambda_2 v_1 v_2 & & & \lambda_{\rho-1}\lambda_\rho v_{\rho-1}v_\rho \\ \vdots & & \ddots & \\ \lambda_1\lambda_\rho v_1 v_\rho & \cdots & \lambda_{\rho-1}\lambda_\rho v_{\rho-1}v_\rho & \sum_{i\neq\rho} \lambda_i^2 \|\hat{v}_i\|^2 \end{bmatrix}$$

We recall the expansion of a determinant in terms of principal minors:

<u>Lemma 5</u>.

Let $A \in M_{\rho\times\rho}(R)$, for some commutative ring R, and write $A = A' + A''$, where $A' = \operatorname{diag}\{a_{11},\ldots,a_{\rho\rho}\}$. If $1 \leq i_1 < \ldots < i_k \leq \rho$, write $A_{i_1\ldots i_k}$ for the principal minor of A obtained by deleting all rows and columns except $i_1,\ldots,i_k$.

In the permutation group S_ρ, write $S_{\rho,k}$ for the subgroup of all *shuffles* of length k (cf. [Spi] vol. 1, Ch. 7). Then:

$$\det A = \sum_{k=1}^{\rho} \sum_{\mu \in S_{\rho,k}} A'_{\mu(1)\ldots\mu(k)} A''_{\mu(k+1)\ldots\mu(\rho)}. \qquad \square$$

Corollary 3.

Let $A_1, \ldots, A_\rho, w_1, \ldots w_\rho \in R$. Then:

$$\begin{vmatrix} A_1 & w_1w_2 & \cdots & w_1w_\rho \\ w_1w_2 & \ddots & & \vdots \\ \vdots & & \ddots & w_{\rho-1}w_\rho \\ w_1w_\rho & \cdots & w_{\rho-1}w_\rho & A_\rho \end{vmatrix} = A_1 \ldots A_\rho + \sum_{k=2}^{\rho} (-1)^{k-1}(k-1) \sum_{1 \leq i_1 < \ldots < i_k \leq \rho} A_1 \ldots \hat{A}_{i_1} \ldots \hat{A}_{i_k} \ldots A_\rho \, w_{i_1}^2 \ldots w_{i_k}^2$$

$\square$

Corollary 4.

In the polynomial ring $\mathbb{Z}[x_1, \ldots, x_\rho]$, write $X_i = x_1 + \ldots + \hat{x}_i + \ldots + x_\rho$. Then:

$$X_1 \ldots X_\rho = \sum_{k=2}^{\rho} (k-1) \sum_{1 \leq i_1 < \ldots < i_k \leq \rho} X_1 \ldots \hat{X}_{i_1} \ldots \hat{X}_{i_k} \ldots X_\rho \, x_{i_1} \ldots x_{i_k}$$

Proof. Put $x_i = w_i^2$. Then by Corollary 3:

$$(-1)^\rho X_1 \ldots X_\rho + \sum_{k=2}^{\rho} (-1)^{\rho-k}(-1)^{k-1}(k-1) \sum_{1 \leq i_1 < \ldots < i_k \leq \rho} X_1 \ldots \hat{X}_{i_1} \ldots \hat{X}_{i_k} \ldots X_\rho \, x_{i_1} \ldots x_{i_k}$$

$$= \begin{vmatrix} -X_1 & w_1w_2 & \cdots\cdots & w_1w_\rho \\ w_1w_2 & \ddots & & \vdots \\ \vdots & & \ddots & w_{\rho-1}w_\rho \\ w_1w_\rho \cdots & w_{\rho-1}w_\rho & & -X_\rho \end{vmatrix} = 0 \; .$$

□

Now, let $\mathcal{S}_r = \{\phi \in C^2(M,N) : \text{rank } \phi > r\}$, an open submanifold of $C^2(M,N)$.

Theorem 2.

τ_2 is elliptic on $\mathcal{S}_2$.

Proof. Write

$$w_i = \lambda_i v_i, a_i = \lambda_i^2 \|\hat{v}_i\|^2 \text{, and } A_i = a_1 + \ldots + \hat{a}_i + \ldots + a_\rho .$$

Then:

$$A(\phi,v) = \begin{bmatrix} A_1 & w_1w_2 & \cdots\cdot & w_1w_\rho \\ w_1w_2 & \ddots & & \vdots \\ \vdots & & \ddots & w_{\rho-1}w_\rho \\ w_1w_\rho & \cdots\; w_{\rho-1}w_\rho & & A_\rho \end{bmatrix}$$

so that, by Corollary 3:

$$\det A(\phi,v) = A_1 \ldots A_\rho - \sum_{i<j} A_1 \ldots \hat{A}_i \ldots \hat{A}_j \ldots A_\rho \, w_i^2 \, w_j^2$$

$$+ 2 \sum_{i<j<k} A_1 \ldots \hat{A}_i \ldots \hat{A}_j \ldots \hat{A}_k \ldots A_\rho \, w_i^2 w_j^2 w_k^2 - \ldots + (-1)^{\rho-1}(\rho-1) w_1^2 \ldots w_\rho^2$$

If $i \neq j$, then $v_i^2\, v_j^2 \leq \|\hat{v}_i\|^2\ \|\hat{v}_j\|^2$, with equality precisely when $v = (0,\dots,v_i,\dots,v_j,\dots,0)$. Thus, $w_i^2 w_j^2 \leq a_i a_j$, so that:

$$\det A(\phi,v) \geq A_1 \dots A_\rho - \sum_{i<j} A_1 \dots \hat{A}_i \dots \hat{A}_j \dots A_\rho\ a_i a_j$$

$$+ 2 \sum_{i<j<k} A_1 \dots \hat{A}_i \dots \hat{A}_j \dots \hat{A}_k \dots A_\rho\ w_i^2 w_j^2 w_k^2 - \dots + (-1)^{\rho-1}(\rho-1) w_1^2 \dots w_\rho^2$$

with strict inequality for every $v \in \mathbb{R}^m \setminus \{0\}$ precisely when rank $d\phi(x) \geq 3$. Now, by Corollary 4:

$$\det A(\phi,v) \geq \sum_{k=3}^{\rho} (k-1) \sum_{i_1<\dots<i_k} A_1 \dots \hat{A}_{i_1} \dots \hat{A}_{i_k} \dots A_\rho\ a_{i_1} \dots a_{i_k}$$

$$+ \sum_{k=3}^{\rho} (-1)^{k-1}(k-1) \sum_{i_1<\dots<i_k} A_1 \dots \hat{A}_{i_1} \dots \hat{A}_{i_k} \dots A_\rho\, w_{i_1}^2 \dots w_{i_k}^2$$

$$\geq 0, \text{ since } a_{i_1} \dots a_{i_k} \geq w_{i_1}^2 \dots w_{i_k}^2 .$$

Thus, $\det A(\phi,v) > 0$ for all $v \in \mathbb{R}^m \setminus \{0\}$ iff rank $d\phi(x) \geq 3$ so that, $\tau_2(\phi) = 0$ is elliptic iff rank $\phi > 2$. □

<u>Conjecture</u>

τ_r is elliptic on $\mathcal{S}_r$. □

Presume only difficulty is the complicated calculation

§7. THE SECOND VARIATION OF $E_r(\phi_t, g)$

By way of comparison with [Smi], we calculate the Hessian of the r-th energy, and the corresponding Jacobi operator.

The r-th Newton tensor is a positive operator (Remark 5.1, Lemma 4.4). So, as tensors in $\phi^{-1}(TN)$, let us define the r-*th Laplacian* of ϕ:

$$\Delta_{\phi,r}(v) = \text{Trace}\ \nabla^2 v(\chi_{r-1}(\phi)^{\frac{1}{2}}, \chi_{r-1}(\phi)^{\frac{1}{2}}) \in \mathbb{C}(\phi^{-1}(TN))$$

and the r-*the Ricci tensors* of ϕ:

$$\text{Ric}_{\phi,r}(v) = \text{Trace}\ R^N(v, d\phi \circ \chi_{r-1}(\phi)^{\frac{1}{2}}) d\phi \circ \chi_{r-1}(\phi)^{\frac{1}{2}} \in \mathbb{C}(\phi^{-1}(TN))$$

$$\text{Ric}_{\phi,r}(v,w) = h(\text{Ric}_{\phi,r}(v), w) \in \mathbb{C}(M \times \mathbb{R})$$

for all $v,w \in \mathbb{C}(\phi^{-1}(TN))$. Now, suppose that ϕ is $(r+1)$-harmonic, and let Φ be a 2-parameter variation of ϕ with variation fields $\frac{\partial\Phi}{\partial s} = v$ and $\frac{\partial\Phi}{\partial t} = w$. Let $H_{\phi,r+1}(v,w)$ denote the *Hessian* of $E_{r+1}(\phi)$ (evaluated at (v,w)). Then:

Theorem 1.

$$H_{\phi,r+1}(v,w) = 2\int_M \{h(\nabla v \circ \chi_r(\phi)^{\frac{1}{2}}, \nabla w \circ \chi_r(\phi)^{\frac{1}{2}}) - \text{Ric}_{\phi,r+1}(v,w)\} v_g$$

$$+ \sum_{p+q+u=r-1} (-1)^u \int_M f_u(\phi)\{\text{Trace}[H_v \circ \chi_p(\phi)]\,\text{Trace}[H_w \circ \chi_q(\phi)]$$

$$- \text{Trace}\,[H_v \circ \chi_p(\phi) \circ H_w \circ \chi_q(\phi)]\} v_g$$

where

$$H_v = g^{-1} \left.\frac{\partial\phi^*_{s,t}h}{\partial s}\right|_{(0,0)} \in \mathbb{C}(T^*M \otimes TM) \quad \text{(cf. Proposition 3.3).}$$

Proof.

$$H_{\phi,r+1}(v,w) = \left.\frac{\partial^2 E_{r+1}(\phi_{s,t})}{\partial t\,\partial s}\right|_{(0,0)} = \int_M \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} \sigma_{r+1}(\phi_{s,t}) v_g.$$

By repeated application of Lemma 4.5, bearing in mind that p16

Trace $= \sigma_1$:

$$\frac{\partial^2}{\partial t\partial s}\sigma_{r+1}(\phi_{s,t}) = \frac{\partial}{\partial t}\text{Trace}\,\{g^{-1}\frac{\partial\phi^*_{s,t}h}{\partial s}\chi_r(\phi_{s,t})\}$$

$$= \text{Trace}\ g^{-1}\,\{\frac{\partial^2\phi^*_{s,t}h}{\partial t\ \partial s}\chi_r(\phi) + \frac{\partial\phi^*_{s,t}h}{\partial s}\ \frac{\partial\chi_r(\phi_{s,t})}{\partial t}\}$$

(1) Let $\{E_i\}$ be a g-orthonormal frame diagonalising $g^{-1}\phi^*h$. Then, by Proposition 3.4: p10

$$\frac{1}{2}\text{Trace}\,\{g^{-1}\frac{\partial^2\phi^*_{s,t}h}{\partial t\ \partial s}\chi_r(\phi)\} = \frac{1}{2}\sum_i(\frac{\partial^2\phi^*_{s,t}h}{\partial t\ \partial s})_{ii}\,\chi_r(\phi)^i_i$$

$$= \sum_i\{h(R^N(w,d\phi(E_i))v,d\phi(E_i)) + h(\nabla_{E_i}\nabla_{\frac{\partial}{\partial t}}v,\ d\phi(E_i))$$

$$+ h(\nabla_{E_i}v,\ \nabla_{E_i}w)\}\chi_r(\phi)^i_i$$

$$= \text{Ric}_{\phi,r+1}(v,w)+h(\nabla v\circ\chi_r(\phi)^{\frac{1}{2}},\nabla w\circ\chi_r(\phi)^{\frac{1}{2}})+h(\nabla\nabla_{\frac{\partial}{\partial t}}v,\ d\phi\circ\chi_r(\phi)).$$

On integrating by parts, the third summand vanishes, since ϕ is (r+1)-harmonic.

(2) By Lemma 4.7 and Lemma 4.5: p16

$$\frac{\partial}{\partial t}\chi_r(\phi_{s,t}) = \sum_{p+q+u=r-1}(-1)^u f_u(\phi)\{\frac{\partial}{\partial t}\sigma_{p+1}(\phi_{s,t})\chi_q(\phi) - \chi_q(\phi)\circ H_w\circ\chi_p(\phi)\}$$

$$= \sum_{p+q+u=r-1}(-1)^u f_u(\phi)\{\text{Trace}[H_w\circ\chi_p(\phi)]\chi_q(\phi)-\chi_q(\phi)\circ H_w\circ\chi_p(\phi)\}$$

$$\mathrm{Trace}\ \{g^{-1}\ \frac{\partial\phi^*_{s,t}h}{\partial s}\ \frac{\partial\chi_r(\phi_{s,t})}{\partial t}\} = \mathrm{Trace}\ \{H_v \circ \frac{\partial\chi_r(\phi_{s,t})}{\partial t}\}$$

$$= \sum_{p+q+u=r-1} (-1)^u f_u(\phi)\{\mathrm{Trace}[H_w\circ\chi_p(\phi)]\ \mathrm{Trace}\ [H_v\circ\chi_q(\phi)]$$

$$- \mathrm{Trace}\ [H_v \circ \chi_q(\phi) \circ H_w \circ \chi_p(\phi)]\}. \quad \square$$

Theorem 1 shows that $H_{\phi,r}(v,w)$ is symmetric, but does not readily reveal the corresponding Jacobi operator. For this, we approach the calculation of $H_{\phi,r}$ from a slightly different angle:

<u>Theorem 2</u>.

$$H_{\phi,r+1}(v,w) = -2 \int_M h(\mathrm{Ric}_{\phi,r+1}(w)$$

$$+ \Delta_{\phi,r+1}(w) + \mathrm{div}(d\phi \circ \frac{\partial}{\partial t}\chi_r(\phi_{0,t})),v)v_g$$

<u>Proof</u>. By Proposition 3.1:

$$\frac{\partial}{\partial s}\Big|_{(0,t)} E_{r+1}(\phi_{s,t}) = -2 \int_M h(\tau_{r+1}(\phi_{0,t}),v_{0,t})v_g.$$

Thus, by the (r+1)-harmonicity of ϕ:

$$\frac{\partial^2}{\partial t\partial s}\Big|_{(0,0)} E_{r+1}(\phi_{s,t}) = -2 \int_M h\ (\frac{\partial}{\partial t}\tau_{r+1}(\phi_{0,t}),v)v_g.$$

Let $\{E_i\}$ be a g-orthonormal frame diagonalising $g^{-1}\phi^*h$, chosen normal for convenience. Then:

$$\frac{d}{dt}\tau_{r+1}(\phi_t) = \frac{d}{dt}\,\text{Trace}\,\nabla(d\phi_t \circ \chi_r(\phi_t))$$

$$= \frac{d}{dt}\sum_i \nabla_{E_i}(d\phi_t \circ \chi_r(\phi_t))(E_i)$$

$$= \frac{d}{dt}\sum_i \{\nabla_{E_i} d\phi_t(\chi_r(\phi_t)(E_i)) + d\phi_t\,\nabla_{E_i}\chi_r(\phi_t)(E_i)\}$$

$$= \sum_i \{\nabla_{\frac{\partial}{\partial t}} \nabla_{\bar{E}_i}\, d\Phi(\chi_r(\phi)(E_i),0) + \nabla_{E_i} d\Phi\ (\frac{d}{dt}\chi_r(\phi_t)(E_i))$$

$$+ \nabla_{\frac{\partial}{\partial t}}\, d\Phi(\nabla_{E_i}\chi_r(\phi),0) + d\phi\nabla_{E_i}\frac{d}{dt}\chi_r(\phi_t)(E_i)\}$$

$$= \sum_i \{\nabla_{\bar{E}_i}\nabla_{\frac{\partial}{\partial t}}\, d\Phi(\chi_r(\phi)(E_i),0) + R^N(w,d\phi(E_i))d\phi(\chi_r(\phi)(E_i))$$

$$+ \nabla_{\frac{\partial}{\partial t}}\, d\Phi(\nabla_{E_i}\chi_r(\phi)(E_i),0) + \nabla_{E_i}(d\phi \circ \frac{d}{dt}\chi_r(\phi_t))(E_i)\}$$

noting that $[\overline{TR}, \overline{TM}] = 0$, and applying Proposition 1.1.

$$= \text{Ric}_{\phi,r+1}(w) + \text{Trace}\,\nabla(d\phi \circ \frac{d}{dt}\chi_r(\phi_t))$$

$$+ \sum_i \{\nabla_{E_i}\nabla_{\chi_r(\phi)E_i} w + \nabla w(\nabla_{E_i}\chi_r(\phi)(E_i))\}$$

by Proposition 2.1 (cf. the proof of Proposition 3.3).

$$= \text{Ric}_{\phi,r+1}(w) + \Delta_{\phi,r+1}(w) + \text{div}\,(d\phi \circ \frac{d}{dt}\chi_r(\phi_t)). \quad \square$$

Let us define the r^{th} *Jacobi operator* of ϕ (on sections of $\phi^{-1}(TN)$) by:

$$J_{\phi,r}(v) = \text{Ric}_{\phi,r}(v) + \Delta_{\phi,r}(v) + \text{div}(d\phi \circ \frac{d}{dt}\chi_{r-1}(\phi_t))$$

for any variation ϕ_t of ϕ with variation field v. By Theorem 1, $J_{\phi,r}$ is self-adjoint w.r.t. h.

<u>Remark</u>. When r = 1, the divergence term vanishes, so that $J_{\phi,1}$ is indeed the Jacobi operator for a harmonic map ϕ (cf. [Smi]).

In the light of Theorem 6.2, we expect $J_{\phi,r}$ to be elliptic precisely when rank $\phi > r$, at least in the case $r = 2$. The equation $J_{\phi,r}(v) = 0$ determines a *semi-linear* 2nd order system of n equations, which we write in the form (cf. §6): ρ33

$$J_{\phi,r}(v)^{\gamma} = P_{\gamma\beta}(\partial)v^{\beta} + F_{\gamma}(v) = 0$$

where $F_{\gamma}(v)$ is of first order. Working with the same coordinates as in §6, we have:

<u>Proposition</u>.

If rank $d\phi(x) = \rho$, the, for any $X \in \mathbb{R}^m$, the symbol of $J_{\phi,2}$ evaluated at X is:

$$P_{\gamma\beta}(X) = \begin{array}{cc} \left(\begin{array}{c|c} A(X) & 0 \\ \hline 0 & D(X) \end{array}\right) & \begin{array}{c} \rho \\ n-\rho \end{array} \\ \begin{array}{cc} \rho \quad & \quad n-\rho \end{array} & \end{array}$$

where:

$$D(X) = \operatorname{diag}\{\sum_i \sigma_{1,i}(\phi)X_i^2, \quad , \sum_i \sigma_{1,i}(\phi)X_i^2\}$$

$$A(X) = \begin{bmatrix} \sum_{i\neq 1} \lambda_i^2 \|\hat{X}_i\|^2 & \lambda_1\lambda_2 X_1 X_2 & \cdots\cdots & \lambda_1\lambda_\rho X_1 X_\rho \\ \lambda_1\lambda_2 X_1 X_2 & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_{\rho-1}\lambda_\rho X_{\rho-1} X_\rho \\ \lambda_1\lambda_\rho X_1 X_\rho & \cdots\cdots & \lambda_{\rho-1}\lambda_\rho X_{\rho-1} X_\rho & \sum_{i\neq\rho} \lambda_i^2 \|\hat{X}_i\|^2 \end{bmatrix}$$

$(\hat{X}_i = (X_1,\ldots,\hat{X}_i,\ldots,X_m))$.

Proof. Terms of 2^{nd} order in the v^γ arise from the divergence term, and 2^{nd} Laplacian.

(1) $\Delta_{\phi,r}(v) = \text{Trace } \nabla^2 v(\chi_{r-1}(\phi)^{\frac{1}{2}}, \chi_{r-1}(\phi)^{\frac{1}{2}}) = \sum_i \nabla_{\partial_i} \nabla v(\partial_i)\sigma_{r-1,i}(\phi)$

$$\nabla v = (v_i^\gamma + v^\alpha \phi_i^\beta \; {}^N\Gamma^\gamma_{\alpha\beta})dx^i \otimes \frac{\partial}{\partial y^\gamma}$$

$$\nabla_{\partial_j} \nabla v = \frac{\partial^2 v^\gamma}{\partial x^i \partial x^j} dx^i \otimes \frac{\partial}{\partial y^\gamma} + \text{lower order terms}$$

$$\Delta_{\phi,r}(v)^\gamma = \sum_i \sigma_{r-1,i}(\phi) \frac{\partial^2 v^\gamma}{\partial x^i \partial x^i} + \text{lower order terms.}$$

(2) From Lemma 4.6 and Proposition 6.1, we have:

$$\operatorname{div}(d\phi \circ \frac{d}{dt}\chi_1(\phi_t)) = \operatorname{div}\,(d\phi \circ (2\langle d\phi,\nabla v\rangle - H_V))$$

$$= \sum_i \nabla_{\partial_i} [d\phi \circ (2\langle d\phi,\nabla v\rangle - H_V)](\partial_i)$$

$$= 2\, d\phi \text{ grad } \langle d\phi,\nabla v\rangle - d\phi \operatorname{div} H_V + \text{lower order terms.}$$

(a) $\langle d\phi,\nabla v\rangle = g^{ij}\ \phi_i^\alpha(v_j^\beta + v^\gamma\phi_j^\delta\ \Gamma_{\gamma\delta}^\beta)h_{\alpha\beta}$

$$\text{grad}\ \langle d\phi,\nabla v\rangle = \sum_{i,j,\beta} \phi_i^\beta \frac{\partial^2 v^\beta}{\partial x^i \partial x^j}\frac{\partial}{\partial x^j} + \text{lower order terms}$$

$$(d\phi\ \text{grad}\ \langle d\phi,\nabla v\rangle)^\gamma = \sum_{i,j,\beta} \phi_j^\gamma\ \phi_i^\beta \frac{\partial^2 v^\beta}{\partial x^i \partial x^j} + \text{lower order terms.}$$

$$= \sum_{\beta=1}^m \lambda_\gamma\lambda_\beta \frac{\partial^2 v^\beta}{\partial x^\beta \partial x^\gamma} + \text{lower order terms.}$$

(b) By Proposition 3.3:

$$H_v = g^{jk}\{h(d\phi(\partial_i),\nabla_{\partial_k} v) + h(d\phi(\partial_k),\nabla_{\partial_i} v)\}\ dx^i \otimes \frac{\partial}{\partial x^j}$$

$$= g^{jk}\{\phi_i^\alpha(v_k^\beta + v^\gamma\phi_k^\delta\Gamma_{\gamma\delta}^\beta) + \phi_k^\alpha(v_i^\beta + v^\gamma\ \phi_i^\delta\Gamma_{\gamma\delta}^\beta)\}h_{\alpha\beta}\ dx^i \otimes \frac{\partial}{\partial x^j}$$

$$\nabla_{\partial_k} H_v = \sum_\beta \{\phi_i^\beta\frac{\partial^2 v^\beta}{\partial x^k \partial x^j} + \phi_j^\beta \frac{\partial^2 v^\beta}{\partial x^k \partial x^i}\}\ dx^i \otimes \frac{\partial}{\partial x^j} + \text{lower order terms}$$

$$\text{div}\ H_v = \sum_k \nabla_{\partial_k} H_v(\partial_k) = \sum_{i,\beta} \{\phi_i^\beta \frac{\partial^2 v^\beta}{\partial x^i \partial x^j} + \phi_j^\beta \frac{\partial^2 v^\beta}{\partial x^i \partial x^i}\}\frac{\partial}{\partial x^j} + \text{lower order terms}$$

$$(d\phi\ \text{div}\ H_v)^\gamma = \sum_{i,j,\beta} \{\phi_j^\gamma\phi_i^\beta \frac{\partial^2 v^\beta}{\partial x^i \partial x^j} + \phi_j^\gamma\phi_j^\beta \frac{\partial^2 v^\beta}{\partial x^i \partial x^i}\} + \text{lower order terms}$$

$$= \sum_{\beta=1}^m \{\lambda_\gamma\lambda_\beta \frac{\partial^2 v^\beta}{\partial x^\beta \partial x^\gamma} + \delta_{\gamma\beta} \sum_{i=1}^m \lambda_\gamma^2 \frac{\partial^2 v^\gamma}{\partial x^i \partial x^i}\} + \text{lower order terms.}$$

Now, for any $X \in \mathbb{R}^m$ and $\gamma,\beta < \rho$:

$$A_{\gamma\beta}(X) = \lambda_\gamma\lambda_\beta X_\gamma X_\beta \qquad (\gamma \neq \beta)$$

$$A_{\gamma\gamma}(X) = \sum_i (\sigma_{1,i}(\phi) - \lambda_\gamma^2)X_i^2 + \lambda_\gamma^2 X_\gamma^2\} = \sum_{i\neq\gamma} \lambda_i^2\ \|\hat{X}_i\|^2. \quad \square$$

We saw in the proof of Theorem 6.2 that:

$$\det A(X) > 0 \text{ for all } X \in \mathbb{R}^m \setminus \{0\} \text{ iff } \operatorname{rank} d\phi(x) \geq 3.$$

Thus:

Theorem 3.

$J_{\phi,2}$ is elliptic iff rank $\phi > 2$. □

Conjecture.

If $r > 2$, then $A(X)$ and $D(X)$ have the same form as their counterparts in §6, and:

$J_{\phi,r}$ is elliptic iff rank $\phi > r$. □

§8. THE FIRST VARIATION OF $Er(\phi,g_t)$ - STRESS-ENERGY TENSORS

Let g_t be a variation of the Riemannian metric g, with variation field κ. Then, for small enough t, g_t is also a Riemannian metric, and κ_t non-degenerate.

Lemma 1.

For all vector fields X, 1-forms θ, and sufficiently small s:

$$g_s(\kappa_s^{-1}(\theta),X) = -g_s(\theta,\kappa_s(X)).$$

Proof. Since $g_t^{-1}g_t = 1_{TM}$ for small t, we have:

$$\left.\frac{dg_t^{-1}}{dt}\right|_s g_s = -g_s^{-1} \left.\frac{dg_t}{dt}\right|_s \text{ i.e. } \kappa_s^{-1} \circ g_s = -g_s^{-1} \circ \kappa_s.$$

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By Lemma 5.1,

$$g_s(\theta,\kappa_s(X)) = g_s(g_s^{-1}\theta, g_s^{-1} \circ \kappa_s(X))$$

$= \kappa(g^{-1}\theta, X) = \kappa(X, g^{-1}\theta) =$

$$= g_s(g_s^{-1} \circ \kappa_s \circ g_s^{-1}(\theta), X) = -g_s(\kappa_s^{-1}(\theta), X). \qquad \square$$

Proposition 1

$$\frac{d}{dt}\Big|_s E_r(\phi,g_t) = \int_M g_s\left(\tfrac{1}{2}\sigma_r(\phi,g_s)g_s - \phi^*h\circ\chi_{r-1}(\phi,g_s),\kappa_s\right)v_{g_s}.$$

Proof.

$$\frac{d}{dt}\Big|_s E_r(\phi,g_t) = \int_M \frac{d}{dt}\Big|_s \sigma_r(\phi,g_t)v_{g_s} + \int_M \sigma_r(\phi,g_s)\frac{d}{dt}\Big|_s v_{g_t}.$$

Let $\{E_i\}_1^m$ be a g_s-orthonormal frame. Then:

(a) $\frac{d}{dt}\Big|_s \sigma_r(\phi,g_t) = \text{Trace}\,[\kappa_s^{-1} \circ \phi^*h \circ \chi_{r-1}(\phi,g_s)]$, by Lemma 4.5

$$= \sum_i g_s(\kappa_s^{-1} \circ \phi^*h \circ \chi_{r-1}(\phi,g_s)(E_i),E_i)$$

$= -\sum_i g_s(\phi^*h\circ\chi_{r-1}(\phi,g_s)(E_i),\kappa_s(E_i))$, by Lemma 1

$$= -g_s(\phi^*h \circ \chi_{r-1}(\phi,g_s),\kappa_s).$$

(b) $\frac{d}{dt}\Big|_s v_{g_t} = \frac{d}{dt}\Big|_s \det(g_s^{-1}g_t)^{\frac{1}{2}}v_{g_s}$, by Lemma 5.2

$$= \tfrac{1}{2}\det(g_s^{-1}g_s)^{-\frac{1}{2}}\frac{d}{dt}\Big|_s \det(g_s^{-1}g_t)v_{g_s}$$

$= \frac{1}{2}$ Trace $[g_s^{-1} \circ \kappa_s \circ \chi_{m-1}(1_{TM})]v_{g_s}$, by Lemma 4.5, noting that det $= \sigma_m$

$= \frac{1}{2} \sum_i g_s(g_s^{-1} \circ \kappa_s(E_i), E_i)v_{g_s}$, since $\chi_{m-1}(1_{TM}) = 1_{TM}$

$= \frac{1}{2} g_s(\kappa_s, g_s)v_{g_s}$. □

Define the r^{th} *stress-energy tensor* of (ϕ, g) by:

$$S_r(\phi) = S_r(\phi, g) = \frac{1}{2}\sigma_r(\phi)g - \phi^*h \circ \chi_{r-1}(\phi).$$

Remark 1. $S_1(\phi, g) = e(\phi)g - \phi^*h$, the stress-energy tensor of ϕ (cf. [B-E]).

Theorem 1.

g is a critical point of $E_r(\phi, g_t)$ iff $S_r(\phi, g) = 0$. □

Remark 2. Lowering an index and plugging-in the recurrence relation of Lemma 4.3, we have that:

$$g^{-1}S_r(\phi) = \chi_r(\phi) - \frac{1}{2}\sigma_r(\phi).$$

Corollary 1.

If rank $\phi < r$, then $S_r(\phi, g) = 0$ for any metric g.

Proof. If rank $\phi < r$, then $\sigma_r(\phi, g) = 0$ for any g (Proposition 5.2) so that $g^{-1}\phi^*h$ has at most $(r-1)$ non-zero eigenvalues. But then all the eigenvalues of $\chi_r(\phi)$ vanish (Lemma 4.4), so that $S_r(\phi) = 0$, By Remark 2. □

The vanishing of $S_r(\phi, g)$ is completely characterized in the following:

<u>Proposition 2</u> (cf. [B-E] Example 3.3)

$S_r(\phi,g) = 0$ iff ϕ is r-conformal and either

(i) rank $\phi < r$

or (ii) $m = 2r$.

<u>Proof</u>. Let $\{E_i\}$ be a g-orthonormal frame diagonalising $g^{-1}\phi^*h$, and put $\lambda_i = \|d\phi(E_i)\|$.

(1) <u>$S_r(\phi,g) = 0 \Rightarrow \phi$ is r-conformal</u>

If $S_r(\phi,g) = 0$, then (by Remark 2) all the eigenvalues of $\chi_r(\phi)$ are the same; $\sigma_{r,i}(\phi) = \frac{1}{2}\sigma_r(\phi) = \sigma_{r,j}(\phi)$ for all i,j (Lemma 4.4). By Proposition 4.2: p14

$$\sigma_{r,i}(\phi) = \lambda_j^2\,\sigma_{r-1,i,j}(\phi) + \sigma_{r,i,j}(\phi), \text{ so that}$$

$$\sigma_{r,i}(\phi) = \sigma_{r,j}(\phi) \text{ iff } (\lambda_i^2 - \lambda_j^2)\sigma_{r-1,i,j}(\phi) = 0.$$

The following possibilities exist:

(a) Rank $d\phi(x) = m$. Then $\sigma_{r-1,i,j}(\phi) \neq 0$ so that $\lambda_i(x) = \lambda_j(x) = \lambda(x)$ say, for all i,j; thus $\phi^*h(x) = \lambda^2(x)g(x)$.

(b) Rank $d\phi(x) < m$. Say that $\lambda_i(x) = 0$. Then:

$$0 = \sum_j \lambda_j^2(x)\sigma_{r-1,i,j}(\phi)(x) = r\sigma_{r,i}(\phi)(x) = \frac{r}{2}\,\sigma_r(\phi)(x)$$

so that rank $d\phi(x) < r$ (Proposition 5.2).

<u>Note</u>. $r\,\sigma_r(\phi) = \sum_i \lambda_i^2\,\sigma_{r-1,i}(\phi)$ (a particular case of an identity in the elementary symmetric polynomials). Thus, by Proposition 4.2 (iii)

$$r\sigma_{r,i}(\phi) = \sum_{j\neq i} \lambda_j^2\,\sigma_{r-1,i,j}(\phi).$$

Consequently, ϕ is r-conformal.

(2) <u>$S_r(\phi,g) = 0$ and rank $\phi \geq r \Rightarrow m = 2r$</u>

If $S_r(\phi,g) = 0$, then:

$$0 = \text{Trace } g^{-1}S_r(\phi) = \text{Trace } \chi_r(\phi) - \frac{m}{2}\sigma_r(\phi)$$

$$= (m-r)\sigma_r(\phi) - \frac{m}{2}\sigma_r(\phi), \text{ by Proposition 4.3}$$

$$= \frac{1}{2}(m-2r)\sigma_r(\phi).$$

So, if $\sigma_r(\phi)$ does not vanish identically (Proposition 5.2), then $m = 2r$.

(3) <u>ϕ r-conformal and $m = 2r \Rightarrow S_r(\phi,g) = 0$</u>

There are two possibilities to consider:

(a) Rank $d\phi(x) < r$. Then, by Corollary 1, $S_r(\phi,g)(x) = 0$.

(b) Rank $d\phi(x) = m$. Then $\lambda_1(x) = \ldots = \lambda_m(x) = \lambda(x)$, say, so that:

$$\chi_r(\phi,g)(x) = \binom{m-1}{r}\lambda(x)^{2r}\,\theta^i \otimes E_i$$

$$\sigma_r(\phi,g)(x) = \binom{m}{r}\lambda(x)^{2r}\theta^i \otimes E_i$$

$$g^{-1}S_r(\phi,g)(x) = \left\{\binom{m-1}{r} - \frac{1}{2}\binom{m}{r}\right\}\lambda(x)^{2r}\,\theta^i \otimes E_i$$

$$= \frac{1}{2m}\binom{m}{r}(m-2r)\,\lambda(x)^{2r}\,\theta^i \otimes E_i = 0. \qquad \square$$

We now consider those (ϕ,g) which are critical for both $E_r(\phi_t,g)$ and $E_r(\phi,g_t)$. For this, we make use of variations of ϕ which fix $\phi(M)$ i.e. those variation fields v which factor through $d\phi$:

$$\begin{tikzcd} TM \arrow[r, "d\phi"] \arrow[d] & TN \arrow[d] \\ M \arrow[u, bend left, "X"] \arrow[r, "\phi"'] \arrow[ur, "v"] & N \end{tikzcd}$$

<u>Remark 4</u>. If X has 1-parameter group $\{\xi_t\}$, then $d\phi(X)$ is the variation field for $\phi_t = \phi \circ \xi_t$.

<u>Definition</u>. If $U \subset M$, write $\int_U \sigma_r(\phi,g)v_g = E_r(\phi,g,U)$.

<u>Proposition 3</u>.

Suppose that $r > 1$. If $S_r(\phi,g) = 0 = \tau_r(\phi,g)$, then ϕ is homothetic and $m = 2r$, or rank $\phi < r$.

<u>Proof</u>. Assume rank $\phi \geq r$. Then by Proposition 2, $m = 2r$ and ϕ is r-conformal. Thus, there is a smooth $\lambda: M \to \mathbb{R}$ s.t.:

$$\phi^*h(x) = \lambda^2(x), \text{ for } x \in \text{supp } \lambda.$$

$$\sigma_r(\phi)(x) = 0, \text{ for } x \in M \setminus \text{supp } \lambda.$$

Write $\bar{U} = \text{supp } \lambda$. Then, for any variation ϕ_t of ϕ:

$$E_r(\phi_t,g) = E_r(\phi_t,g,U) + E_r(\phi_t,g,M\setminus U)$$

$$= E_r(\phi_t,\lambda^2 g,U) + E_r(\phi_t,g,M\setminus U),$$

by the conformal invariance of $E_r(\phi_t)$ (Proposition 5.3).

Thus,

$$\frac{d}{dt}\Big|_0 E_r(\phi_t,g) = \frac{d}{dt}\Big|_0 E_r(\phi_t,\lambda^2 g,U) + \frac{d}{dt}\Big|_0 E_r(\phi_t,g,M\setminus U)$$

$$= \frac{d}{dt}\Big|_0 E_r(\phi_t,\lambda^2 g,U), \text{ since } E_r(\phi,g,M\setminus U) = 0.$$

is an absolute minimum

Now, suppose that $v = d\phi(X)$, for some $X \in \mathcal{C}(TM)$. Then, by Lemma 6.2:

$$\frac{d}{dt}\Big|_o Er(\phi_t,\lambda^2 g,U) = 2\int_U \langle d\phi \circ \chi_{r-1}(\phi), \nabla(d\phi \circ X)\rangle v_{\lambda^2 g}$$

If $\{E_i\}$ is a g-orthonormal frame, then $\{\frac{1}{\lambda} E_i\}$ is λ^2g-orthonormal, and:

$$\langle d\phi\circ\chi_{r-1}(\phi),\nabla(d\phi\circ X)\rangle = \sum_i \frac{1}{\lambda^2} h(d\phi\circ\chi_{r-1}(\phi)(E_i),\nabla_{E_i}(d\phi\circ X)).$$

Since $\phi:(U,\lambda^2 g) \to (N,h)$ is an isometric immersion, $\nabla d\phi$ is normal-valued (see Corollary 2.5.1). (Note: We are now working with the Riemannian connection of $(M,\lambda^2 g)$). Thus:

reference number ?

$$\langle d\phi\circ\chi_{r-1}(\phi), \nabla(d\phi\circ X)) = \sum_i \frac{1}{\lambda^2} h(d\phi\circ\chi_{r-1}(\phi)(E_i), d\phi(\nabla_{E_i} X))$$

$$= \sum_i \frac{1}{\lambda^2} \phi^*h(\chi_{r-1}(\phi)(E_i),\nabla_{E_i} X)$$

$$= \sum_i \frac{1}{\lambda^2} g(g^{-1}\phi^*h \circ \chi_{r-1}(\phi)(E_i), \nabla_{E_i} X), \text{ by Lemma 5.1}$$

$$= \frac{1}{2}\frac{1}{\lambda^2} g(\sigma_r(\phi)E_i, \nabla_{E_i} X), \text{ since } S_r(\phi,g) = 0,$$

$$= \frac{1}{2}(\lambda^2 g)(\frac{\sigma_r(\phi)}{\lambda^2}, \nabla X).$$

Integrating by parts (Lemma 6.1):

$$\frac{d}{dt}\Big|_o E_r(\phi_t,\lambda^2 g,U) = -\int_U (\lambda^2 g)(\text{grad}_{\lambda^2 g}(\frac{\sigma_r(\phi)}{\lambda^2}),X)v_{\lambda^2 g}.$$

Now, $\tau_r(\phi,g) = 0 \Rightarrow \mathrm{grad}_{\lambda^2 g}\left(\frac{\sigma_r(\phi)}{\lambda^2}\right) = 0$, by Theorem 5.1

$$\Rightarrow \sigma_r(\phi) = \text{const. } \lambda^2 \Rightarrow \lambda^{2r} = \text{const. } \lambda^2$$

$\Rightarrow \lambda = \text{const.}$, since by supposition $r > 1$. □

Remark 5. The situation is strikingly different when $r = 1$, in which case (Remark 5.3, Proposition 2) the hypothesis of Proposition 3 is that we have a weakly conformal harmonic map of a Riemann surface. It has recently become known how to construct and classify all such maps into $\mathbb{C}P^n$ (for example) which in addition are *isotropic* (and full) (cf [E-W]). In a particular instance of this, Chern had previously contructed minimal immersions $S^2 \to S^4$ inducing metrics of non-constant Gaussian curvature on S^2 (and hence certainly not homothetic) ([Che]).

In the equidimensional case, we have the converse:

Corollary 2.

Suppose that $\dim N = \dim M$. Then:

(i) If $r = 1$; $S(\phi,g) = 0 \Rightarrow \tau(\phi,g) = 0$

(ii) If $r > 1$; $S_r(\phi,g) = 0 = \tau_r(\phi,g) \iff \phi$ is homothetic and $m = 2r$, or rank $\phi < r$.

Proof. The additional requirement $m = n$ means that *every* variation field may be factored through $d\phi$ over U. Thus, rewriting the last few lines in the proof of Proposition 3:

$$\tau_r(\phi,g) = 0 \text{ iff } \operatorname{grad}\left(\frac{\sigma_r(\phi)}{\lambda^2}\right) = 0 \text{ iff } \lambda^{2r} = \text{const. } \lambda^2$$

$$\text{iff } \lambda = \text{const., and } r > 1; \text{ or } r = 1. \qquad \square$$

<u>Remark 6.</u> Recalling Remark 5.3 and Proposition 2, Corollary 2(i) reads:

"Every weakly conformal map $(M^2,g) \to (N^2,h)$ is harmonic",

which is Proposition 2.3 of [Lem].

We can enlarge the set of critical points of $Er(\phi,g_t)$ by restricting the class of variation of g to those *generated by vector fields;* viz. if $X \in \mathfrak{C}(TM)$ with 1-parameter group $\{\xi_t\}$, we consider only those variations $g_t = \xi_t^* g$.

<u>Remark 7.</u> The variation field for such a variation is $\kappa_s = L_X g_s$.

<u>Lemma 2.</u>

With respect to the Riemannian connection of (M,g):

$$L_X g = 2 \operatorname{Sym} \nabla(gX), \text{ for all } X \in \mathfrak{C}(TM)$$

where gX denotes the covariant representation of X (cf. §5).

<u>Proof.</u> Since $(gX)Y = g(X,Y)$ for all $X,Y \in \mathfrak{C}(TM)$, we have that:

$$\nabla_Y(gX)(Z) = Y.g(X,Z) - g(X,\nabla_Y Z) = g(\nabla_Y X,Z).$$

Now,

$$L_X g(Y,Z) = X.g(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]),$$

since L_X is a derivation of $\mathfrak{T}(M)$.

$$= X.g(Y,Z) - g(\nabla_Y X - \nabla_X Y, Z) - g(Y, \nabla_Z X - \nabla_X Z)$$

$$= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = \nabla_Y(gX)(Z) + \nabla_Z(gX)(Y). \quad \square$$

Proposition 4.

If g_t is a variation of g generated by a vector field X, then:

$$\frac{d}{dt}\Big|_s E_r(\phi, g_t) = -2 \int_M g_s(\operatorname{div} S_r(\phi, g_s), g_s X) v_{g_s}.$$

Proof. Let $\{E_i\}$ be a g_s-orthonormal frame which is ϕ^*h-orthogonal; then $\{E_i\}$ is also $S_r(\phi, g_s)$-orthogonal. By Proposition 1 and Remark 7:

$$\frac{d}{dt}\Big|_s E_r(\phi, g_t) = \int_M g_s(S_r(\phi, g_s), L_X g_s) v_{g_s}$$

$$= \sum_i \int_M S_r(\phi, g_s)(E_i, E_i) L_X g_s(E_i, E_i) v_{g_s}$$

$$= \sum_i \int_M S_r(\phi, g_s)(E_i, E_i) \nabla_{E_i}(g_s X)(E_i) v_{g_s}, \text{ by Lemma 3}$$

$$= 2 \int_M g_s(S_r(\phi, g_s), \nabla(g_s X)) v_{g_s}$$

$$= -2 \int_M g_s (\operatorname{div} S_r(\phi, g_s), g_s X) v_{g_s}, \text{ by Lemma 6.1.} \quad \square$$

By the Riesz Lemma for Hilbert spaces:

Theorem 2.

g is a critical point of $E_r(\phi, g_t)$ w.r.t. variations generated by vector fields iff $\operatorname{div} S_r(\phi,g) = 0$. □

By looking at simultaneous variations of both ϕ and g generated by vector fields (Remarks 4 and 7) we are able to see how the "restricted" critical points of $E_r(\phi,g_t)$ relate to the critical points of $E_r(\phi_t,g)$.

Lemma 3.

If (ϕ_t,g_t) is a variation of (ϕ,g) generated by a vector field with 1-parameter group $\{\xi_t\}$, then:

$$\sigma_r(\phi_t,g_t) = \sigma_r(\phi,g) \circ \xi_t.$$

Proof. If $\{E_i\}$ is a g-orthonormal frame which is ϕ^*h-orthogonal, then $\{d\xi_t^{-1}(E_i)\}$ is a ξ_t^*g-orthonormal frame which is ϕ_t^*h-orthogonal. Thus, by Remark 5.1:

$$\text{Spectrum of } g_t^{-1}\phi_t^*h = \{\|d\phi_t(d\xi_t^{-1}(E_i))\|^2\} = \{\|d\phi(E_i)\|^2\}$$

$$= \text{Spectrum of } g^{-1}\phi^*h$$

so that the higher-power energy densities of (ϕ_t,g_t) and (ϕ,g) coincide. □

Lemma 4.

If ξ is an orientation-preserving diffeomorphism of M, then:

$$v_{\xi^*g} = \xi^* v_g.$$

Proof. If $\{E_i\}$ is a positively oriented g-orthonormal frame, then $\{d\xi^{-1}(E_i)\}$ is a positively oriented ξ^*g-orthonormal frame, so that:

$$v_{\xi^* g}(d\xi^{-1}(E_1),\ldots,d\xi^{-1}(E_m)) = 1 = v_g(E_1,\ldots,E_m)$$

$$= \xi^* v_g(d\xi^{-1}(E_1),\ldots,d\xi^{-1}(E_m)). \quad \square$$

<u>Lemma 5</u>.

If ω is an m-form on M, and X a vector field, then:

$$\int_M L_X\omega = 0.$$

<u>Proof</u>. (cf. [K-N] vol 1, Appendix 6).

An application of Stokes' Theorem, bearing in mind that M is compact, oriented, and boundaryless. $\square$

<u>Lemma 6</u>.

For variations generated by vector fields:

$$\frac{d}{dt}\Big|_o E_r(\phi_t,g_t) = 0.$$

<u>Proof</u>. By Lemmas 3 and 4:

$$\frac{d}{dt}\Big|_o E_r(\phi_t,g_t) = \int_M \frac{d}{dt}\Big|_o \sigma_r(\phi_t,g_t)v_{g_t}$$

$$= \int_M \frac{d}{dt}\Big|_o (\sigma_r(\phi,g) \circ \xi_t.\ \xi_t^* v_g)$$

$$= \int_M \frac{d}{dt}\Big|_o \xi_t^*(\sigma_r(\phi,g)v_g) = \int_M L_X(\sigma_r(\phi,g)v_g) = 0, \text{ by Lemma 5.} \quad \square$$

<u>Remark 8</u>. Lemma 6 holds for any Lagrangian $\mathcal{L}(\phi,g)$ satisfying Lemma 3.

Theorem 3.

$$\operatorname{div} S_r(\phi,g) = -h(\tau_r(\phi,g),d\phi).$$

Proof. For any variations ϕ_t of ϕ and g_t of g, we have:

$$\frac{d}{dt} E_r(\phi_t,g_t) = \frac{d}{dt} E_r(\phi_t,g) + \frac{d}{dt} E_r(\phi,g_t).$$

When ϕ_t and g_t are generated by a vector field X, we have (Proposition 6.1, Proposition 4, Lemma 6, Remark 4):

$$0 = -2\int_M h(\tau_r(\phi,g),d\phi(X))v_g - 2\int_M g(\operatorname{div} S_r(\phi,g),gX)v_g$$

$$= -2\int_M \{h(\tau_r(\phi,g),d\phi) + \operatorname{div} S_r(\phi,g)\}(X)v_g, \text{ for all } X.$$

We note that for any 1-form θ on M, $\theta(X) = g(\theta,gX)$ so that putting $X = g^{-1}\ \{h(\tau_r(\phi,g),d\phi) + \operatorname{div} S_r(\phi,g)\}$ gives the result. □

Corollary 3.

$$\operatorname{div} \chi_r(\phi,g) = \frac{1}{2}\, d\sigma_r(\phi,g) - h(\tau_r(\phi,g),d\phi).$$

Proof. By Remark 2, $g^{-1}S_r(\phi,g) = \chi_r(\phi,g) - \frac{1}{2}\sigma_r(\phi,g)$. □

Remark 9. In general, and in contrast to those considered by R. Reilly, the Newton tensors of ϕ are not divergence-free. However, if ϕ is a *Riemannian immersion* (which is the assumption throughout [Rei]) we have that $\sigma_r(\phi,g) = \binom{m}{r}$, so that $d\sigma_r(\phi,g) = 0$. Moreover, the $\tau_r(\phi,g)$ are all proportional and, in particular, normal to $\phi(M)$, so that $h(\tau_r(\phi,g),d\phi) = 0$. So, in this case, the $\chi_r(\phi,g)$ are indeed divergence-free.

Corollary 4. (cf. [B-E] Theorem 2.9).

If $\tau_r(\phi,g) = 0$, then $\operatorname{div} S_r(\phi,g) = 0$.

Conversely, if ϕ is a *submersion* and $\operatorname{div} S_r(\phi,g) = 0$, then $\tau_r(\phi,g) = 0$. $\square$

We now deduce the following generalisation of Proposition 3:

Theorem 4.

Suppose that $(m,r) \neq (2,1)$, and ϕ is r-conformal and r-harmonic. Then, either

ϕ is homothetic, or rank $\phi < r$.

Proof. If $m = 2r$, then r-conformality implies that $S_r(\phi,g) = 0$ (Proposition 2) and the conclusion follows from Proposition 3.

So, assume now that $m \neq 2r$, and rank $\phi \not< r$. Then, the r-conformality of ϕ means that $\phi^*h = \lambda^2 g$ on some (non-empty) open set. In part (3) of the proof of Proposition 2 we calculated:

$$g^{-1}S_r(\phi,g) = \frac{1}{2m}\binom{m}{r}(m-2r)\lambda^{2r}\,\theta^i \otimes E_i$$

for any g-orthonormal frame $\{E_i\}$ diagonalising $g^{-1}\phi^*h$. Thus:

$$g^{-1}\operatorname{div} S_r(\phi,g) = \operatorname{div} g^{-1}S_r(\phi,g) = \sum_i \nabla_{E_i} g^{-1}S_r(\phi,g)(E_i)$$

$$= \frac{1}{2m}\binom{m}{r}(m-2r)\operatorname{grad}\lambda^{2r}.$$

Now, By Corollary 4:

$$\tau_r(\phi,g) = 0 \Rightarrow \operatorname{div} S_r(\phi,g) = 0$$

$$\Longleftrightarrow \operatorname{grad}\lambda^{2r} = 0 \text{ (since } m \neq 2r) \quad \Longleftrightarrow \lambda = \text{const.} \qquad \square$$

We conclude by featuring the stress-energy tensor in a swift calculation of the Euler-Lagrange equations for the volume functional (see §5).

Let

$$U_\phi = \{x \in M : \text{rank } d\phi(x) = m\}.$$

Then, U_ϕ is an open submanifold of M with Riemannian metric ϕ^*h, over which:

$$\begin{aligned} \text{vol}(\phi)\ v_g &= \det\,(g^{-1}\phi^*h)^{\frac{1}{2}} v_g = v_{\phi^*h}, \text{ by Lemma 5.2} \\ &= \det\,((\phi^*h)^{-1}\phi^*h)\, v_{\phi^*h} = \sigma_m(\phi,\phi^*h)\, v_{\phi^*h} \end{aligned}$$

Of course, vol $(\phi) = 0$ over $M\backslash U_\phi$ (see Remark 5.2).

Remark 10. Vol (ϕ) is independent of the metric on M, which sheds some light on the condition required for equality in Proposition 5.4.

Proposition 5.

If ϕ_t is a variation of ϕ, with variation field v, then:

$$\frac{d}{dt}\Big|_0 \text{Vol } (\phi_t) = -\int_{U_\phi} h(\tau(\phi,\phi^*h),v)\, v_{\phi^*h}.$$

Proof. For sufficiently small t we have that $U_\phi \subset U_{\phi_t}$. Write:

$$\begin{aligned} \text{Vol } (\phi_t) &= \int_{U_\phi} \sigma_m(\phi_t,\phi_t^*h) v_{\phi_t^*h} + \int_{M\backslash U_\phi} \sigma_m(\phi_t,\phi_t^*h)\, v_{\phi_t^*h} \\ &= E_m(\phi_t,\phi_t^*h,U_\phi) + E_m(\phi_t,\phi_t^*h,M\backslash U_\phi). \end{aligned}$$

Then,

$$\frac{d}{dt}\Big|_o \mathrm{Vol}\ (\phi_t) = \frac{d}{dt}\Big|_o E_m(\phi_t,\phi_t^*h,U_\phi),\ \text{since } E_m(\phi,\phi^*h,M\backslash U_\phi) = 0.$$

$$= \frac{d}{dt}\Big|_o E_m(\phi_t,\phi^*h,U_\phi) + \frac{d}{dt}\Big|_o E_m(\phi,\phi_t^*h,U_\phi)$$

$$= -2\int_{U_\phi} h(\tau_m(\phi,\phi^*h),v)v_{\phi^*h} + \int_{U_\phi} \phi^*h(S_m(\phi,\phi^*h),\ \frac{d\phi_t^*h}{dt}\Big|_o)v_{\phi^*h}$$

by Proposition 1 and Proposition 6.1.

Now,

$$(\phi^*h)^{-1}S_m(\phi,\phi^*h) = \chi_m(\phi,\phi^*h) - \frac{1}{2}\sigma_m(\phi,\phi^*h) = -\frac{1}{2}1_{TM},$$

since, by the Cayley-Hamilton Theorem, $\chi_m(\phi) = 0$ (Remark 4.2).

Thus,

$$\phi^*h(S_m(\phi,\phi^*h),\ \frac{d\phi_t^*h}{dt}\Big|_o) = -\frac{1}{2}\phi^*h(\phi^*h,\ \frac{d\phi_t^*h}{dt}\Big|_o)$$

$$= -\frac{1}{2}\,\mathrm{Trace}\ (\phi^*h)^{-1}\circ\frac{d\phi_t^*h}{dt}\Big|_o$$

$$= -\frac{1}{2}\frac{d}{dt}\Big|_o \sigma_1(\phi_t,\phi^*h),\ \text{by Lemma 4.5} = -\langle d\phi,\nabla v\rangle,$$

by Lemma 6.2.

Also,

$$\tau_m(\phi,\phi^*h) = \mathrm{Trace}\ \nabla(d\phi\circ\chi_{m-1}(\phi,\phi^*h))$$

$$= \mathrm{Trace}\ \nabla(d\phi\circ 1_{TM}) = \tau(\phi,\phi^*h)$$

referring back to part (3) in the proof of Proposition 2.

Thus,

$$\frac{d}{dt}\Big|_o \text{Vol}(\phi_t) = -2\int_{U_\phi} h(\tau(\phi,\phi^*h),v)v_{\phi^*h} - \int_{U_\phi} \langle d\phi,\nabla v\rangle v_{\phi^*h}$$

$$= -\int_{U_\phi} h(\tau(\phi,\phi^*h),v)v_{\phi^*h},\ \text{on integrating by parts}$$

(Lemma 6.1). □

Theorem 5.

ϕ is a critical point of the volume functional iff $\tau(\phi,\phi^*h) \restriction U_\phi = 0$. □

CHAPTER 2 : FIBRE BUNDLE GEOMETRY

§1. GENERALITIES ([Eel], [Hus], [K-N] vol 1, [Spi] vol 2)

Let $\xi: P \to M$ be a principal G-bundle, and Q a left G-manifold, for some Lie group G. Form the associated (G,Q)-bundle $\sigma: P\times_G Q = E \to M$:

$$\begin{array}{ccc} P & \xleftarrow{\pi_P} & P \times Q \\ {\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ M & \xleftarrow[\sigma]{} & E \end{array}$$

where μ is the quotient of $P \times Q$ by the right G-action:

$$(p,q).g = (p.g,\ g^{-1}q).$$

Choosing any $p \in P$, let ξ_p (resp. μ_p) denote the corresponding diffeomorphism of G (resp. Q) with the fibre of ξ (resp. σ) over $\xi(p)$:

$$\xi_p : G \to P_{\xi(p)};\ \xi_p(g) = g.p$$

$$\mu_p : Q \to E_{\xi(p)};\ \mu_p(q) = \mu(p,q).$$

When ξ is endowed with a connection, the splitting $TP = (TP)^V \oplus (TP)^H$ gives rise to a decomposition $T(P \times Q) = \overline{TP}^V \oplus \overline{TP}^H \oplus \overline{TQ}$, where, for example, $\overline{TP}^H_{(p,q)} = di_q(p)(T_pP)^H$. It is easily seen that $d\mu(\overline{TQ}) = (TE)^V$ and $d\mu(\overline{TP})^V = 0$; in addition, the G-*invariance* of $(TP)^H$ means there is a well-defined distribution $d\mu(\overline{TP})^H$ on E. Thus, TE also acquires a splitting:

$$TE = (TE)^V \oplus (TE)^H = d\mu(\overline{TP})^V \oplus d\mu(\overline{TP})^H$$

which we refer to as an *(associated) connection* in E. The inter-relationships of these various distributions is summarised in the following picture:

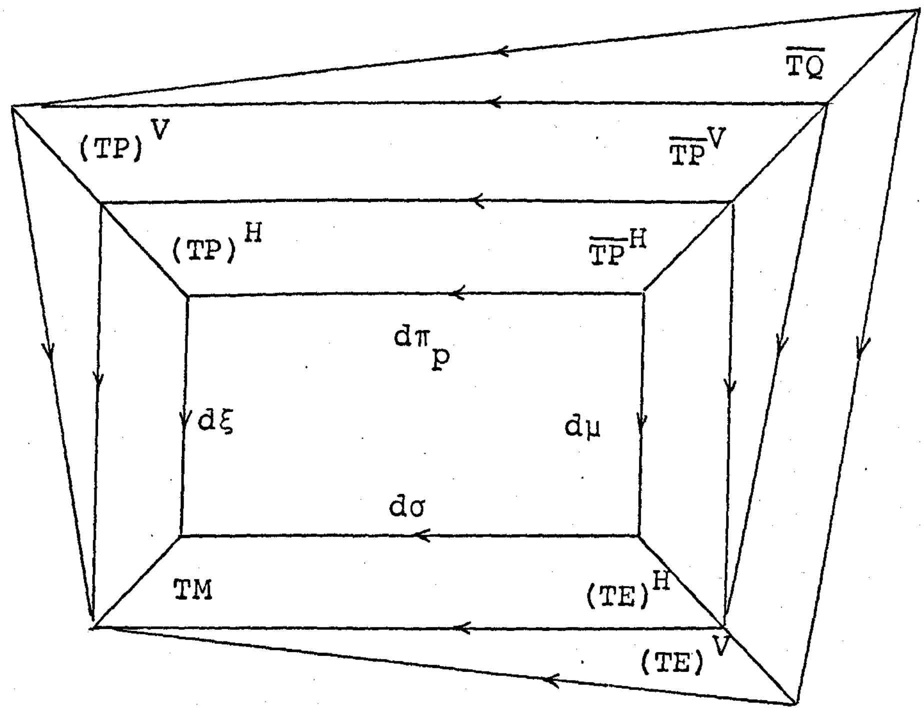

FIGURE 1.

The "unique global horizontal path lifting" property of connections in P carries over to associated connections (and characterises the latter - see [Ehr]). Indeed, if $p_\gamma(t)$ denotes the *horizontal lift* of a path $\gamma(t)$ in M with $p_\gamma(0) = p \in P_{\gamma(0)}$, and if $e = \mu(p,q)$ $(\in E_{\gamma(0)})$, then $e_\gamma(t) = \mu(p_\gamma(t),q)$ is the horizontal lift of $\gamma(t)$ to E, through e. The horizontal lifts of $\gamma(t)$ to P (resp. E) are the trajectories of a horizontal field on the submanifold $\coprod_{t\in\mathbb{R}} \xi^{-1}(\gamma(t)) \subset P$ (resp. $\coprod_{t\in\mathbb{R}} \sigma^{-1}(\gamma(t)) \subset E$),

the flow for which is *parallel translation* along γ:

$$\xi_t^{\gamma}(p) = p_{\gamma}(t), \text{ for all } p \in P_{\gamma(0)}$$

$$\sigma_t^{\gamma}(e) = e_{\gamma}(t) = \mu(p_{\gamma}(t),q) = \mu(\xi_t^{\gamma}(p),q) = \mu_{\xi_t^{\gamma}(p)} \circ \mu_p^{-1}(e),$$

for *any* $p \in P_{\sigma(e)}$.

If (M,g), (Q,ℓ) and $(G,\langle,\rangle)$ are Riemannian manifolds, with the metrics on G and Q being G-invariant, then by a combination of horizontally lifting g, and transferring ℓ and $\langle,\rangle$ to fibres via the maps ξ_p and μ_p, each of P, P × Q, and E may be given a natural Riemannian metric. It is clear that w.r.t. these metrics all bundle projections become Riemannian submersions; moreover, they all have totally geodesic fibres ([Vil] Theorem 3.5). Concerning the Levi-Civita connection of (E,h):

<u>Proposition</u>

Let τ_E, τ_Q, and τ_M be the tangent bundles of E, Q and M resp. Suppose that $e(t)$, $f(t)$ are paths in E with $e(0) = f(0)$, and let $\sigma\circ e(t) = \gamma(t)$ and $\sigma\circ f(t) = \delta(t)$.

(i) If $e(t)$ is horizontal and $f(t)$ is vertical, then:

$$(\tau_E)_t^e\,(f'(0)) = \frac{d}{ds}\Big|_0\ \sigma_t^{\gamma}(f(s))$$

(ii) If $e(t)$ is vertical and $f(t)$ is horizontal, then:

$$(\tau_E)_t^e\,(f'(0)) = \frac{d}{ds}\Big|_0\ \sigma_s^{\delta}(e(t)).$$

(iii) If both e(t) and f(t) are horizontal, then:

$$d\sigma(\tau_E)_t^e\,(f'(0)) = (\tau_M)_t^\gamma\,(d\sigma(f'(0))).$$

(iv) If both e(t) and f(t) are vertical, then:

$$(\tau_E)_t^e\,(f'(0)) = d\mu_p(\tau_Q)_t^q(r'(0))$$

where $p \in P_{\gamma(0)}$, $e(t) = \mu(p,q(t))$, and $f(t) = \mu(p,r(t))$.

Proof.

Check that the connection characterised by these parallel translations is metric and torsion-free. □

§2. HORIZONTAL/VERTICAL DECOMPOSITION OF PATHS

Let e(t) be a path in E, with $\sigma \circ e(t) = \gamma(t)$. We may "decompose" e(t) into a horizontal path:

$$e^H(t) = e_\gamma(t) = \sigma_t^\gamma(e(0)),$$

and a vertical path:

$$e^V(t) = (\sigma_t^\gamma)^{-1}(e(t)).$$

Their germs provide a decomposition of the tangent vector e'(0) into its horizontal and vertical components:

Proposition.

(i) $(e^H)'(0) = e'(0)^H$.

(ii) $(e^V)'(0) = e'(0)^V$.

Proof. Firstly, we show this to be the case for a path p(t) in the principal G-bundle ξ. Let g(t) denote the path in G s.t.

$p^H(t) = p(t).g(t)$; viz. $g(t) = \xi^{-1}_{p(t)}(p^H(t))$. Because parallel translation in P is G-equivariant:

$$p(0) = (\xi^\gamma_t)^{-1}(p^H(t)) = (\xi^\gamma_t)^{-1}(p(t).g(t))$$

$$= (\xi^\gamma_t)^{-1}(p(t)).g(t) = p^V(t).g(t)$$

so that $p^V(t) = p(0).g(t)^{-1}$. Since parallel translation is a "flow" ($\xi^\gamma_{s+t} = \xi^\gamma_s \circ \xi^\gamma_t$), $g(t)$ is a 1-parameter subgroup of G. Thus, $g(t) = \exp tX$, for $X = g'(0) \in \mathfrak{g}$. We then have that $g(t)^{-1} = \exp(-tX)$, and:

$$(p^V)'(0) = -X^*(p(0)), \quad (p^H)'(0) = p'(0) + X^*(p(0))$$

where X^* is the *fundamental vector field* on P generated by X; $X^*(p) = \frac{d}{dt}\Big|_0 (p.\exp tX)$. Thus, $(p^V)'(0) + (p^H)'(0) = p'(0)$, with $(p^V)'(0)$ vertical, and $(p^H)'(0)$ horizontal.

To extend to the associated bundle σ, write $e(t) = \mu(p(t),q(t))$. Then

$$e^H(t) = \mu(p^H(t),q(0)), \text{ and } e^V(t) = \mu(p^V(t),q(t)).$$

Thus,

$$(e^H)'(0) = d\mu((p^H)'(0),0), \text{and } (e^V)'(0) = d\mu((p^V)'(0),q'(0)),$$

so that,

$$(e^H)'(0) + (e^V)'(0) = d\mu((p^H)'(0) + (p^V)'(0),q'(0))$$

$$= d\mu(p'(0),q'(0)) = e'(0). \quad \square$$

§3. FIBRE PRODUCTS

If R is another left G-manifold, and $\tau : F = P\times_G R \to M$, we may form the *fibre product* $\rho : E \oplus F \to M$; $(E \oplus F)_x = E_x \times F_x$ for all $x \in M$ (see [Hus], p.15). $E \oplus F$ is isomorphic to the associated bundle $P\times_G(Q \times R)$:

$$\begin{tikzcd}
 & & P\times R \arrow[ddll] \arrow[d, "\eta"] & & \\
 & & F \arrow[ddll, dashed, "\tau"'] & & \\
P \arrow[d, "\xi"'] & & & & P\times Q\times R \arrow[llll] \arrow[uull] \arrow[d] \arrow[ddll] \\
M & & & & E \oplus F \arrow[llll, dashed, "\rho"] \arrow[uull, dashed, "\beta"'] \arrow[ddll, "\alpha"] \\
 & & P\times Q \arrow[uull] \arrow[d, "\mu"] & & \\
 & & E \arrow[uull, "\sigma"] & &
\end{tikzcd}$$

$$P\times_G(Q \times R) \cong E \oplus F;\ \nu(p,q,r) \longleftrightarrow (\mu(p,q),\eta(p,r))$$

Thus, up to isomorphism, $\nu_p = \mu_p \times \nu_p : Q \times R \to (E \oplus F)_{\xi(p)}$.

$E \oplus F$ may also be considered as fibering over E and F, as the associated bundles

$$\alpha : \mathcal{E} = (P \times Q)\times_G R \to E \text{ and } \beta : \mathcal{F} = (P \times R)\times_G Q \to F \quad \text{resp.}$$

We note that $\mathcal{E}\restriction E_x = (E \oplus F)_x = \mathcal{F}\restriction F_x$,for all $x \in M$.

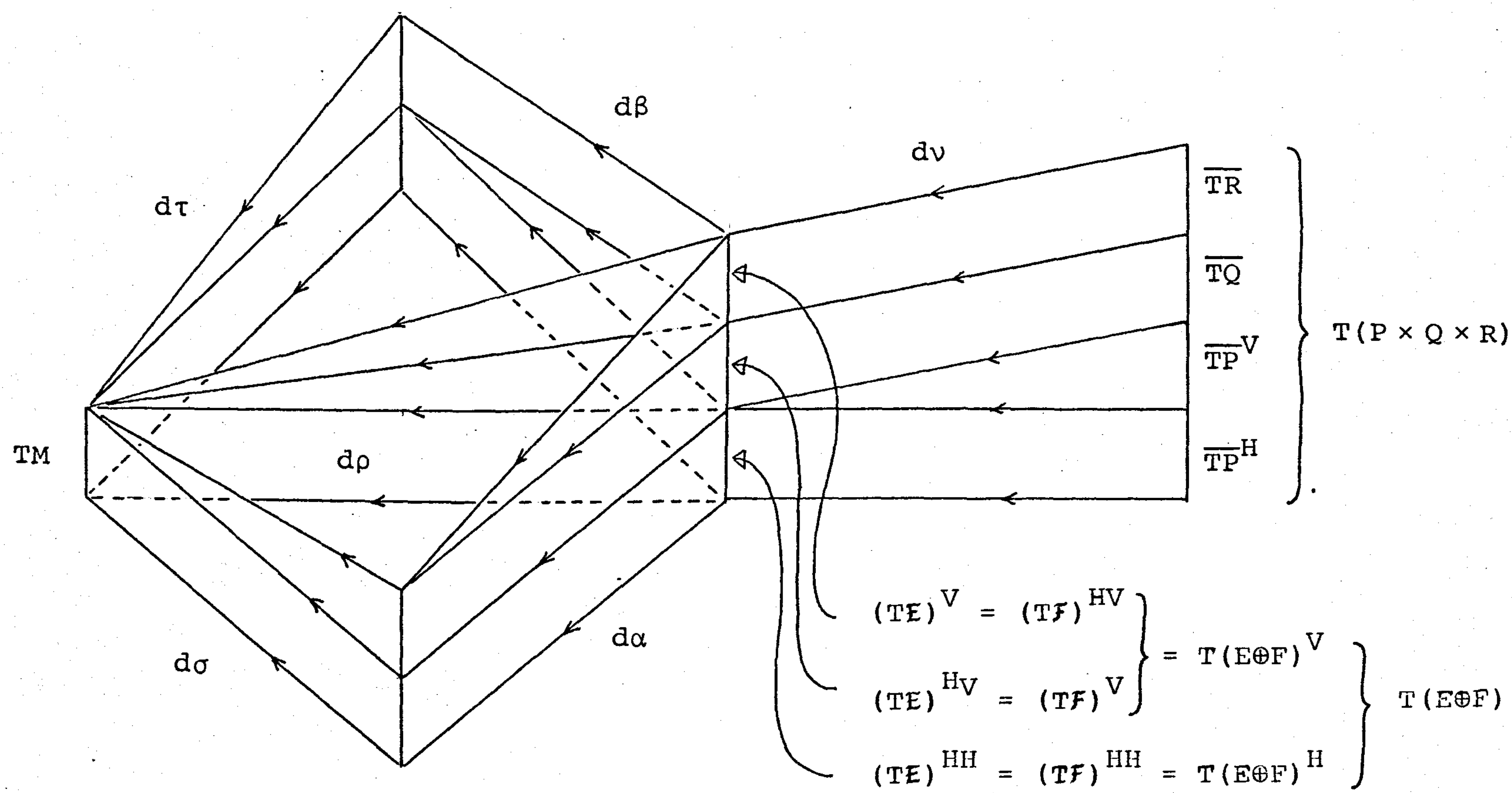
dβ
dν
dτ
$\overline{TR}$
$\overline{TQ}$
$\overline{TP}^V$
$T(P \times Q \times R)$
$\overline{TP}^H$
TM
dρ
$(T\mathcal{E})^V = (T\mathcal{F})^{HV}$
$= T(E \oplus F)^V$
$(T\mathcal{E})^{H_V} = (T\mathcal{F})^V$
$T(E \oplus F)$
dσ
dα
$(T\mathcal{E})^{HH} = (T\mathcal{F})^{HH} = T(E \oplus F)^H$

Figure 2.

A connection in P sets up connections in all other bundles, as in §1. In particular, $E \oplus F$, $\mathcal{E}$, and $\mathcal{F}$ each have associated connections, whose inter-relationships are portrayed in Figure 2. Concerning their parallel translations:

Proposition.

Let $e(t)$ and $f(t)$ be curves in E and F resp. covering a curve $\gamma(t)$ in M.

(i) $\alpha_t^e(e(0),f(0)) = (e(t), \tau_t^\gamma(f(0))$

(ii) $\beta_t^f(e(0),f(0)) = (\sigma_t^\gamma(e(0)), f(t))$

(iii) $\rho_t^\gamma(e(0),f(0)) = (\sigma_t^\gamma(e(0)),\tau_t^\gamma(f(0)))$.

Proof. We prove (i); (ii) and (iii) are almost identical.

(i) We show that $(e(t), \tau_t^\gamma(f(0)))$ is the unique horizontal lift of $e(t)$ to $\mathcal{E}$ with initial point $(e(0), f(0))$. Since $\tau_t^\gamma(f(0)) \in F_{\gamma(t)}$, it is clear that $(e(t),\tau_t^\gamma(f(0)))$ covers $e(t)$. Let $e(t) = \mu(p(t),q(t))$ and $f(t) = \eta(p(t),r(t))$.

As in the proof of Proposition 2.1, write $p^H(t) = p(t).g(t)$. Then:

$$(e(t),\tau_t^\gamma(f(0))) = (\mu(p(t),q(t)), \eta(p^H(t), r(0)))$$

$$= (\mu(p^H(t), g(t)^{-1}.q(t)),\eta(p^H(t),r(0)))$$

$$\longleftrightarrow \nu(p^H(t),g(t)^{-1}.q(t),r(0)).$$

Thus,

$$\frac{d}{dt}(e(t), \tau_t^\gamma(f(0))) = d\nu((p^H)'(t),g(t)^{-1}q'(t)-X^*(q(t)),0)$$

and a glance at Figure 2 shows that this is α-horizontal. □

Corollary

When $e(t)$ (resp. $f(t)$) is horizontal, $\rho_t^\gamma = \alpha_t^e$ (resp. β_t^f). □

§4. HOM-BUNDLES

Suppose that G has representations on vector space V and W; then G has a representation on Hom (V,W) by conjugation:

$$(g.L)(v) = g.(L(g^{-1}.v)).$$

Form the vector bundles $E = P\times_G V$, $F = P\times_G W$, and Hom (E,F); then Hom (E,F) is given the geometry of the isomorphic bundle $P\times_G$ Hom (V,W):

$$\begin{tikzcd}
P\times W \arrow[d, "\eta"'] \arrow[dr] & & \\
F \arrow[dr, dashed, "\tau"] & P \arrow[d, "\xi"] & P\times \mathrm{Hom}\,(V,W) \arrow[l] \arrow[d, "\zeta"] \\
P\times V \arrow[ur] \arrow[d, "\mu"'] & M & P\times_G \mathrm{Hom}\,(V,W) \arrow[l, "\kappa"] \\
E \arrow[ur, "\sigma"'] & &
\end{tikzcd}$$

$P\times_G$ Hom $(V,W) \cong$ Hom (E,F); $\zeta_p(L) \longleftrightarrow \eta_p \circ L \circ \mu_p^{-1}$ for any $p \in P$, $L \in$ Hom (V,W). The parallel translations of σ,τ,κ are related:

Proposition.

$$\kappa_t^\gamma(\mathcal{L}) = \tau_t^\gamma \circ \mathcal{L} \circ (\sigma_t^\gamma)^{-1}$$

for any path $\gamma(t)$ in M, and $\mathcal{L} \in$ Hom $(E,F)_{\gamma(0)}$.

<u>Proof</u>. Let $e = \mu_p(v) \in E_{\gamma(0)}$, and $\mathcal{L} \longleftrightarrow \zeta_p(L)$, for some $p \in P_{\gamma(0)}$, $v \in V$, and $L \in \mathrm{Hom}\,(V,W)$. Making use of the above isomorphism, and characterisation of parallel translation given in §1, we have:

$$\kappa_t^\gamma(\mathcal{L})\sigma_t^\gamma(e) = \zeta_{\xi_t^\gamma(p)}(L)\mu_{\xi_t^\gamma(p)}(v) = \eta_{\xi_t^\gamma(p)} \circ L \circ \mu_{\xi_t^\gamma(p)}^{-1}(\mu_{\xi_t^\gamma(p)}(v))$$

$$= \eta_{\xi_t^\gamma(p)}(Lv) = \eta_{\xi_t^\gamma(p)} \circ \eta_p^{-1}(\mathcal{L}e) = \tau_t^\gamma(\mathcal{L}e). \quad \square$$

§5. PARTIAL CONNECTIONS

A *partial connection* in the principal G-bundle $\xi : P \to M$ is a G-invariant distribution $\mathcal{H}$ on P partially complementary to $(TP)^V$ ([KT1] p. 14), and generalises to associated bundles by the construction of §1. An example is the fibre product $E \oplus F \to M$, where a connection in P gives rise to two complementary partial connections in $\mathcal{E} \to E$; $(T\mathcal{E})^H = (T\mathcal{E})^{HV} \oplus (T\mathcal{E})^{HH}$. (See Figure 2). By the G-invariance of $\mathcal{H}$, we may define the distribution $\Delta = d\xi(\mathcal{H})$ on M, and refer to the partial connection as being Δ-*partial*. In particular, a Δ-partial connection in a vector bundle is one permitting covariant differentiation only along directions in Δ. A very special case is that of a Δ-partial connection D in Δ, when there is a torsion tensor defined on Δ-valued vector fields:

$$T(X,Y) = D_X Y - D_Y X - [X,Y]^\Delta.$$

If (M,g) is a Riemannian manifold:

Lemma.

The only metric, torsion-free, Δ-partial connection in $\Delta \subset TM$ is the Δ-projection of the Levi-Civita connection (restricted to Δ).

Proof. (cf. [K-N] vol. 1, p. 160)

If X,Y,Z are Δ-valued vector fields, the formula:

$$2g(D_X Y,Z) = X.g(Y,Z) + Y.g(Z,X) - Z.g(X,Y)$$

$$+ g([X,Y],Z) + g([Z,X],Y) - ([Y,Z],X)$$

defines a torsion-free , metric, Δ-partial connection D in Δ, and must be satisfied by any such partial connection. $\square$

Partial connections provide a neat method for unifying some well-known results concerning Riemannian immersions and submersions ([K-N] vol 2, p.11; [Her], Proposition 3.1

Proposition.

Let $\phi:(M,g) \to (N,h)$, and put $\Delta = (\text{Ker } d\phi)^{\perp}$. Suppose that $d\phi|\Delta$ is isometric. Then, for any Δ-valued vector fields X,Y:

$$\nabla d\phi(X,Y) = \tilde{\nabla}_X(d\phi \circ Y)^{\perp}$$

where $\tilde{\nabla}$ is the pullback connection in $\phi^{-1}(TN)$, and $^{\perp}$ denotes orthogonal projection of $\phi^{-1}(TN)$ onto $d\phi(TM)^{\perp}$.

Proof. (cf. [K-N] vol. 2, p.11).

We write $\tilde{\nabla}_X(d\phi \circ Y) = \tilde{\nabla}_X(d\phi \circ Y)^{T} + \tilde{\nabla}_X(d\phi \circ Y)^{\perp}$ for the decomposition into $d\phi(TM)$- and $d\phi(TM)^{\perp}$-components, and compare with:

$$\tilde{\nabla}_X(d\phi\circ Y) = d\phi(\nabla_X Y) + \nabla d\phi(X,Y).$$

Write $D_X Y$ for that Δ-valued vector field satisfying:

$$d\phi(D_X Y) = \tilde{\nabla}_X(d\phi\circ Y)^T.$$

It is easily checked that D is a Δ-partial connection in Δ. Moreover:

(a) D is torsion-free Since $d\phi$ has symmetric fundamental form, the $d\phi$-torsion of ${}^N\nabla$ vanishes (Proposition 1.2.1). Thus:

$$\tilde{\nabla}_X(d\phi\circ Y) - \tilde{\nabla}_Y(d\phi\circ X) = d\phi[X,Y]$$

Taking $d\phi(TM)$-components, $D_X Y - D_Y X = [X,Y]^{\Delta}$.

(b) D is metric Let Z be another Δ-valued vector field. Then, since $d\phi|\Delta$ is isometric:

$$\begin{aligned} X.g(Y,Z) &= X.h(d\phi\circ Y, d\phi\circ Z) = h(\tilde{\nabla}_X(d\phi\circ Y), d\phi\circ Z) + h(d\phi\circ Y, \tilde{\nabla}_X(d\phi\circ Z)) \\ &= h(d\phi(D_X Y), d\phi(Z)) + h(d\phi(Y), d\phi(D_X Z)) \\ &= g(D_X Y, Z) + g(Y, D_X Z). \end{aligned}$$

Now, by the lemma, $D_X Y = (\nabla_X Y)^{\Delta}$. Thus, $\tilde{\nabla}_X(d\phi\circ Y)^T = d\phi(\nabla_X Y)$ and so $\tilde{\nabla}_X(d\phi\circ Y)^{\perp} = \nabla d\phi(X,Y)$. $\square$

Corollary.

(i) If ϕ is a *Riemannian immersion*, then $\nabla d\phi(X,Y) = (\tilde{\nabla}_X Y)^{\perp}$ for *any* $X,Y \in \mathcal{C}(TM)$ (identifying a vector with its $d\phi$-image).

(ii) If ϕ is a *Riemannian submersion*, then:

(a) $\nabla d\phi((TM)^H, (TM)^H) = 0$

(b) $\nabla d\phi(di_x, di_x) = -d\phi\nabla di_x$, where $i_x : M_x \hookrightarrow M$.

Proof. (ii) (b) $\nabla d(\phi \circ i_x) = \nabla d\phi(di_x, di_x) + d\phi(\nabla di_x)$.

But, $d(\phi \circ i_x) = d\phi \circ di_x = 0$. $\square$

§6. CONNECTION MAPS

A connection in a vector bundle is completely specified by its *connection map* , described by [GKM] (§2.4) in the case of the tangent bundle. We give a co-ordinate-free description valid for arbitrary vector bundles.

If V is a vector space, then we have the identification:

$$V \times V \cong TV; \ (v,v') \longleftrightarrow \frac{d}{dt}\Big|_0 (v + tv').$$

Thus, the inclusion and projection maps of Chapter 1 §1 may be interpreted:

$$\left.\begin{array}{l} j_v : V \xrightarrow{\cong} T_vV \\ \pi_2 : TV \longrightarrow V \end{array}\right\} \text{ and } \pi_2 \circ j_v = 1_V \, ,$$

where $\pi_2 : V \times V \to V$ is projection onto the second factor.

Lemma.

If $L : V \to W$ is linear, the following diagram commutes:

$$\begin{tikzcd} T_v V \arrow[r, "dL(v)"] \arrow[d, shift right, "\pi_2"'] & T_{L(v)} W \arrow[d, shift left, "\pi_2"] \\ V \arrow[r, "L"'] \arrow[u, shift right, "j_v"'] & W \arrow[u, shift left, "j_{L(v)}"] \end{tikzcd}$$

□

Now, if $\sigma : E \to M$ is the vector bundle associated to the principal G-bundle $\xi : P \to M$ via some representation of G on V, we have the morphisms:

$$\begin{tikzcd} E \arrow[r, "j_\alpha"] \arrow[d, shift left, "\sigma"] & (TE)^V \arrow[d, "d\sigma"] \\ M \arrow[r, "\mathbb{O}_M"'] \arrow[u, shift left, bend left, "\alpha"] & TM \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "\sigma"] & (TE)^V \arrow[l, "\pi_2"'] \arrow[d, "d\sigma"] \\ M & TM \arrow[l, "\tau_M"] \end{tikzcd}$$

with $\pi_2 \circ j_\alpha = id_E$, for any $\alpha \in \mathfrak{C}(E)$. (Here, $\mathbb{O}_M$ denotes the zero-section of TM).

Let ω be the 1-form for a connection in P. If $X \in TP$ and $A \in TV$ (so that $(X,A) \in T(P \times V)$) we define the *connection map* K corresponding to ω:

$$\begin{tikzcd} E \arrow[d, "\sigma"'] & TE \arrow[l, "K"'] \arrow[d, "d\sigma"] \\ M & TM \arrow[l, "\tau_M"] \end{tikzcd} \qquad K\, d\mu(X,A) = \pi_2\, d\mu(\omega(X)^*, A)$$

where $\omega(X)^*$ is the fundamental vector field in P generated by $\omega(X) \in \mathfrak{g}$ (evaluated at the base-point of X).

Remark 1. Since $\omega(X)^*$ is ξ-vertical, $d\mu(\omega(X)^*,A)$ is σ-vertical (see Fig. 1) so that the identification by π_2 makes sense.

Proposition.

K is well-defined, and has the following properties:

(1) $K|(TE)^H = 0$

(2) $K|(TE)^V = \pi_2$

(3) $K \circ d\bar{a} = \bar{a} \circ K$, for any $a \in \mathbb{R}$, where
$$\begin{tikzcd} E \arrow[rr, "\bar{a}"] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$$
; $\bar{a}(e) = ae$.

Proof. If (X,A), $(Y,B) \in T(P \times V)$, then:

$$d\mu(Y,B) = d\mu(X,A) \text{ iff } (Y,B) = dR_g(X,A) + (Z,0)$$

for some $g \in G$ and $Z \in (TP)^V$, where R_g denotes the (right) action of g on $P \times V$. Thus, for K to be well-defined it suffices to show that:

$$K \circ d(\mu \circ R_g) = K \circ d\mu, \text{ for all } g \in G.$$

$$K \circ d(\mu \circ R_g)(X,A) = K \circ d\mu(dR_g(X), dL_g^{-1}(A))$$

$$= \pi_2 \circ d\mu(\omega(dR_g(X))^*, dL_g^{-1}(A))$$

$$= \pi_2 \circ d\mu((\mathrm{Ad}\, g^{-1}.\omega(X))^*, dL_g^{-1}(A)) \quad \text{([K-N] vol.1, p.64).}$$

$$= \pi_2 \circ d\mu(\omega(X)^*.g,\ dL_g^{-1}(A)).$$

(Note: If $\mathfrak{C}(TP)$ is given the right G-action $X.g = dR_g \circ X \circ R_g^{-1}$, then the map $\mathfrak{g} \to \mathfrak{C}(TP)$; $v \mapsto v^*$ is G-equivariant (where G acts on the right of $\mathfrak{g}$ by $g \mapsto \mathrm{Ad}\, g^{-1}$)).

$$= \pi_2 \circ d\mu(dR_g(\omega(X)^*),\ dL_{g^{-1}}(A))$$

$$= \pi_2 \circ d\mu(\omega(X)^*,A) = K \circ d\mu(X,A).$$

(1) From Fig.1 : $d\mu(X,A) \in (TE)^H$ iff $X \in (TP)^H$ and $A = 0$. Thus:

$$K \circ d\mu(X,0) = \pi_2 \circ d\mu(\omega(X)^*,0) = 0, \text{ since } (TP)^H = \operatorname{Ker} \omega.$$

(2) From Fig.1 : $d\mu(X,A) \in (TE)^V$ iff $X \in (TP)^V$

iff $X = v^*$, for some $v \in \mathfrak{g}$.

Thus:

$$K \circ d\mu(v^*,A) = \pi_2 \circ d\mu(\omega(v^*)^*,A)$$

$$= \pi_2 \circ d\mu(v^*,A) \quad \text{([K-N] vol. 1, p.64)}$$

(3) We note that if $e = \mu(p,v)$, then $\bar{a}(e) = \mu(p,av)$, so that $d\bar{a} \circ d\mu(X,A) = d\mu(X,aA)$. Thus:

$$K \circ d\bar{a} \circ d\mu(X,A) = K \circ d\mu(X,aA) = \pi_2 \circ d\mu(\omega(X)^*,\ aA)$$

$$= \pi_2 \circ d\bar{a} \circ d\mu(\omega(X)^*,A) = \bar{a} \circ \pi_2 \circ d\mu(\omega(X)^*,A), \text{ by the Lemma}$$

$$= \bar{a} \circ K \circ d\mu(X,A). \qquad \square$$

Remark 2. Any vector bundle morphism K satisfying properties (1), (2) and (3) serves to define a connection in E.

If $\alpha \in \mathfrak{C}(E)$, and Y tangent to M, the *covariant derivative* $\nabla_Y\alpha$ is defined to be the vertical component of $d\alpha(Y)$, identified to the fibre through α. Thus:

Corollary.

$$\nabla_Y\alpha = K(d\alpha(Y)). \qquad \square$$

CHAPTER 3 : HARMONIC SECTIONS

§1. GRAPHS OF r-HARMONIC MAPS

We have already noted (Chapter 2, §1) that if $\sigma:E \to (M,g)$ is a fibre bundle where structure group and fibre model have invariant metrics, then a connection in σ gives E a natural Riemannian metric w.r.t. which $(TE)^H = (\ker \sigma)^\perp$. For maximum generality let us therefore consider a submersion of Riemannian manifolds $\sigma:(E,h) \to (M,g)$, and write $(TE)^V = \ker d\sigma$, $(TE)^H = (\ker d\sigma)^\perp$; we need not assume that $d\sigma|(TE)^H$ is isometric. The complementary sub-bundles $(TE)^V$ and $(TE)^H$ inherit Riemannian structures $(h^V, {}^V\nabla)$ and $(h^H, {}^H\nabla)$ resp. from that of (TE,h,∇); for example:

$$ {}^V\nabla_Z W = (\nabla_Z W)^V \quad , \quad h^V(Z,W) = h(Z^V,W^V) $$

for all $Z, W \in \mathcal{C}(TE)$. If $\theta:M \to E$, we have its *vertical differential* $(d\theta)^V$, and r^{th} *vertical Newton tensor* $\chi_r^V(\theta) = \chi_r(g^{-1}\theta^*h^V)$.

We begin by considering the trivial fibration $\sigma = \pi_M : (M \times N, g \times h) \to (M,g)$, for some Riemannian manifold (N,h), in which case $(TE)^V = \overline{TN}$, and $(TE)^H = \overline{TM}$. If

$\phi:M \to N$, put $\theta = \Gamma_\phi = \text{graph } \phi: M \to M \times N$; then $\theta^*(g \times h)^V = \phi^*h$,

and $(d\theta)^V = I_\phi \circ d\phi$ (by Proposition 1.2.2(i), putting $P = M$ and $\psi = id_M$).

Lemma 1

$$I_\phi \, \tau_{r+1}(\phi) = \text{Trace } {}^V\nabla((d\theta)^V \circ \chi_r^V(\theta)).$$

Proof. Let X, Y $\in \mathcal{C}(TM)$. Then, by Proposition 1.2.2 (ii):

$$I_\phi \, \nabla_X(d\phi \circ Y) = {}^V\nabla_X(I_\phi \circ d\phi(Y)) = {}^V\nabla_X((d\theta)^V \circ Y).$$

If $\{E_i\}_1^m$ is an orthonormal frame field in M:

$$I_\phi \, \tau_{r+1}(\phi) = \sum_i I_\phi \, \nabla_{E_i}(d\phi \circ \chi_r(\phi))E_i$$

$$= \sum_i I_\phi\{\nabla_{E_i}(d\phi \circ \chi_r(\phi)E_i) - d\phi \circ \chi_r(\phi)(\nabla_{E_i}E_i)\}$$

$$= \sum_i {}^V\nabla_{E_i}((d\theta)^V \circ \chi_r^V(\theta)(E_i)) - (d\theta)^V \circ \chi_r^V(\theta)(\nabla_{E_i}E_i)$$

$$= \text{Trace } {}^V\nabla((d\theta)^V \circ \chi_r^V(\theta)). \qquad \square$$

In view of this, if $\sigma:(E,h) \to (M,g)$ is any submersion, let us define the r^{th} *vertical tension field* of a map $\theta:M \to E$ by:

$$\tau_r^V(\theta) = \tau_r^V(\theta,g) = \text{Trace } {}^V\nabla((d\theta)^V \circ \chi_{r-1}^V(\theta))$$

and say that θ is *vertically r-harmonic* whenever $\tau_r^V(\theta) = 0$. In particular, when θ is a vertically r-harmonic section of σ, call θ a *r-harmonic section*. Since every section of $\pi_M : M \times N \to N$ is the graph of some map $M \to N$, we conclude from Lemma 1:

Theorem 1.

The r-harmonic sections of $\pi_M:(M \times N, g \times h) \to (M,g)$ are precisely the graphs of the r-harmonic maps $(M,g) \to (N,h)$. $\square$

Now, let $\sigma:E \to M$ be a fibre bundle associated to a principal G-bundle $\xi:P \to M$, with fibre Q. We recall ([Eel]) that the sections of σ are in 1-1 correspondence with the G-equivariant maps $P \to Q$; viz. if $\phi:P \to Q$ is equivariant, so is $\Gamma_\phi:P \to P \times Q$, which thereby factors to give a section θ of σ:

$$\begin{tikzcd} P \arrow[d, "\xi"'] \arrow[r, bend left, "\Gamma_\phi"] & P \times Q \arrow[l, "\pi_P"] \arrow[d, "\mu"] \\ M \arrow[r, bend right, "\theta"'] & E \arrow[l, "\sigma"'] \end{tikzcd}$$

Say that ξ has a connection, and (M,g), (Q,ℓ) are Riemannian manifolds, with G-invariant metric ℓ. Then, geometrising this picture as in Chapter 2, §1 we obtain Riemannian manifolds (P,k) and (E,h) whose metrics inter-relate as follows:

<u>Lemma 2</u>.

(i) $\phi^*\ell = (\theta\circ\xi)^*h^V$

(ii) $g^{-1}\theta^*h^V \circ d\xi = d\xi \circ k^{-1}\phi^*\ell$. Thus, $\chi_r^V(\theta)\circ d\xi = d\xi\circ \chi_r(\phi)$.

<u>Proof</u>.

(i) Since h^V is defined by transferring ℓ to the fibres of σ by the maps μ_p $(p \in P)$, we have:

$$(\theta\circ\xi)^*h^V = (\mu\circ\Gamma_\phi)^*h^V = \Gamma_\phi^*\mu^*h^V = \Gamma_\phi^*(k^V \times \ell) = \phi^*\ell.$$

(ii) Let $\{F_\alpha\}_1^p$ be a k-orthonormal local frame field in TP, with $\{F_\alpha\}_1^m$ horizontal and $\{F_\alpha\}_{m+1}^p$ vertical, and put $E_i = d\xi(F_i)$. Let $Z \in \mathfrak{C}(TP)$. By Proposition 1.4.1:

$$g^{-1}\theta^*h^V \circ d\xi(Z) = \sum_i g(g^{-1}\theta^*h^V \circ d\xi(Z), E_i)E_i$$

$$= \sum_i \theta^*h^V(d\xi(Z), E_i)E_i = \sum_\alpha \theta^*h^V(d\xi(Z), d\xi(F_\alpha))d\xi(F_\alpha)$$

$$= \sum_\alpha (\theta\circ\xi)^*h^V(Z, F_\alpha)d\xi(F_\alpha) = \sum_\alpha \phi^*\ell(Z, F_\alpha)d\xi(F_\alpha), \text{ by (i) above}$$

$$= d\xi \circ k^{-1}\phi^*\ell(Z). \quad \square$$

This leads to the following generalisation of Theorem 1:

Theorem 2.

θ is a r-harmonic section of σ iff ϕ is a (equivariant) r-harmonic map.

Proof. Referring to Figure 1:

$$d(\theta\circ\xi)^V = (d\theta)^V \circ \xi$$

$d(\mu\circ\Gamma_\phi)^V = d\mu \circ (d\Gamma_\phi)^V$, where $(d\Gamma_\phi)^V$ is the π_p-vertical differential of Γ_ϕ.

Applying the tensor product rule for covariant differentiation to the bundles

$$\xi^{-1}(TM) \otimes (\theta\circ\xi)^{-1}(TE)^V \text{ and } \Gamma_\phi^{-1}(\overline{TQ}) \otimes (\mu\circ\Gamma_\phi)^{-1}(TE):$$

$$\nabla(d(\mu\circ\Gamma_\phi)^V \circ \chi_r(\phi)) = \nabla(d\mu\circ(d\Gamma_\phi)^V \circ \chi_r(\phi))$$

$$= \nabla d\mu((d\Gamma_\phi)^V \circ \chi_r(\phi), (d\Gamma_\phi)^V \circ\chi_r(\phi)) + d\mu^V \nabla((d\Gamma_\phi)^V \circ \chi_r(\phi))$$

$= d\mu^V \nabla((d\Gamma_\phi)^V \circ \chi_r(\phi))$, by Corollary 2.5.1 (ii), since the π_p-vertical distribution is μ-horizontal.

$$= d\mu\ {}^V\nabla((d\Gamma_\phi)^V \circ \chi_r^V(\Gamma_\phi))$$

$${}^V\nabla(d(\theta\circ\xi)^V \circ \chi_r(\phi)) = {}^V\nabla((d\theta)^V \circ d\xi \circ \chi_r(\phi))$$

$$= {}^V\nabla((d\theta)^V \circ \chi_r^V(\theta) \circ d\xi), \text{ by Lemma 2 (ii)}$$

$$= {}^V\nabla((d\theta)^V \circ \chi_r^V(\theta))(d\xi,d\xi) + (d\theta)^V \circ \chi_r^V(\theta)\ \nabla d\xi.$$

Since $\mu\circ\Gamma_\phi = \theta\circ\xi$, taking traces:

$$d\mu\ \tau_r^V(\Gamma_\phi) = \tau_r^V(\theta) \circ \xi + (d\theta)^V \circ \chi_r^V(\theta)\tau(\xi).$$

Now, a Riemannian submersion with minimal fibres is harmonic ([E-S] Proposition 4 (c)).

In particular, ξ has totally geodesic fibres, so that $\tau(\xi) = 0$. Thus, since the π_p-vertical distribution is μ-horizontal:

θ is a r-harmonic section of σ iff Γ_ϕ is a r-harmonic section of π_p iff ϕ is a r-harmonic map, by Theorem 1. □

§2. VERTICAL ENERGIES

Let $\sigma:(E,h) \to (M,g)$ be a submersion with compact, oriented, and boundaryless base, and let $\theta:M \to E$. We define the *vertical energy density* of θ to be:

$$e^V(\theta) = \frac{1}{2}\,\|(d\theta)^V\|^2$$

and the *vertical energy functional by:*

$$E^V(\theta) = \int_M e^V(\theta) v_g .$$

It is not hard to see (cf. Proposition 1.4.2) that $e^V(\theta) = \frac{1}{2}$ Trace $g^{-1}\theta^* h^V$, whereupon we define the r^{th} *vertical energy density* of θ to be:

$$\sigma_r^V(\theta) = \sigma_r^V(\theta, g) = \sigma_r(g^{-1}\theta^* h^V)$$

and the r^{th} *vertical energy functional by:*

$$E_r^V(\theta) = \int_M \sigma_r^V(\theta) v_g .$$

In order to investigate the critical points of E_r^V we recall the proof of Proposition 1.3.3 where the calculation of a variation field depended on the symmetry of the second fundamental form of the variation. In the present context the appropriate fundamental form is that of the *vertical* differential of the variation, which is no longer symmetric. We therefore start by tackling this problem.

Let $\{E_i\}_1^m$ and $\{F_\alpha\}_1^{m+q}$ be θ-adapted local orthonormal frame fields in TM and TE resp., with $\{F_\alpha\}_1^m$ horizontal, $\{F_\alpha\}_{m+1}^{m+q}$ vertical, and $\{E_i\}$ chosen to be normal about some point $x \in M$, for convenience. Let $\{F_\alpha\}$ (resp. $\{E_i\}$) have dual frame $\{\psi^\alpha\}$ (resp. $\{\omega^i\}$), and let (ψ_α^γ) (resp. (ω_i^k)) be the matrix of *connection 1-forms* for (E,h) (resp. (M,g)) w.r.t. $\{F_\alpha\}$ (resp. $\{E_i\}$):

$$\psi_\alpha^\gamma \otimes F_\gamma = \nabla F_\alpha .$$

Then, the first structural equation of (M,g) may be written ([Spi] vol. 2, p. 245):

$$\psi^\gamma_\alpha \wedge \psi^\alpha = d\psi^\gamma$$

Lemma 1.

The connection form in $\theta^{-1}(TE)$ has matrix $(\theta^*\psi^\gamma_\alpha)$ w.r.t. the frame $\{\theta^{-1}F_\alpha\}$.

Proof.

If we allow $(\tilde{\psi}^\gamma_\alpha)$ to denote the matrix of the pullback connection form w.r.t. $\{\theta^{-1}F_\alpha\}$, then, recalling Chapter 1, §2:

$$\tilde{\psi}^\gamma_\alpha(X)\,\theta^{-1}F_\gamma = (\theta^{-1}\nabla)_X(\theta^{-1}F_\alpha) = \theta^{-1}(\nabla_{d\theta(x)}F_\alpha)$$

$$= \theta^{-1}(\psi^\gamma_\alpha(d\theta(X))F_\gamma) = \theta^*\psi^\gamma_\alpha(X)\,\theta^{-1}F_\gamma$$

for any $X \in \mathfrak{C}(TM)$. □

Lemma 2.

$$\mathrm{Alt}\ {}^V\nabla(d\theta)^V = \sum_{\gamma>m}\ \sum_{\beta\le m} \theta^*(\psi^\gamma_\beta \wedge \psi^\beta)F_\gamma$$

where Alt denotes "Alternation".

Proof. Confusing $\theta^{-1}F_\alpha$ with F_α, we write:

$$(d\theta)^V = \sum_{\gamma>m} A^\gamma_j\, \omega^j \otimes F_\gamma$$

where $A^\gamma_j = \psi^\gamma(d\theta \circ E_j) = \theta^*\psi^\gamma(E_j)$. Then:

$$\nabla_{E_i}(d\theta)^V = \sum_{\gamma>m} \{dA^\gamma_j(E_i)\omega^i \otimes F_\gamma - A^\gamma_j \omega^j_k(E_i)\omega^k \otimes F_\gamma + A^\gamma_j \omega^j \otimes \theta^*\psi^\alpha_\gamma(E_i)F_\alpha$$

by Lemma 1.

$${}^V\nabla_{E_i}(d\theta)^V = \sum_{\alpha,\gamma>m} \{dA^\gamma_j(E_i) - A^\gamma_k \omega^k_j(E_i) + A^\alpha_j \theta^*\psi^\gamma_\alpha(E_i)\}\omega^j \otimes F_\gamma$$

$${}^V\nabla_{E_i}(d\theta)^V(E_j) - {}^V\nabla_{E_j}(d\theta)^V(E_i) = \sum_{\alpha,\gamma>m} \{dA^\gamma_j(E_i) - dA^\gamma_i(E_j)$$

$$+ A^\gamma_k\ \omega^k_i(E_j) - A^\gamma_k\ \omega^k_j(E_i) + A^\alpha_j \theta^*\psi^\gamma_\alpha(E_i) - A^\alpha_i\ \theta^*\psi^\gamma_\alpha(E_j)\}F_\gamma$$

(a) $dA^\gamma_j(E_i) - dA^\gamma_i(E_j) = \nabla_{E_i} A^\gamma_j - \nabla_{E_j} A^\gamma_i$

$$= \nabla_{E_i}(\theta^*\psi^\gamma(E_j)) - \nabla_{E_j}(\theta^*\psi^\gamma(E_i)) = \nabla_{E_i}\ \theta^*\psi^\gamma(E_j) - \nabla_{E_j}\theta^*\psi^\gamma(E_i)$$

$= 2\ d(\theta^*\psi^\gamma)(E_j, E_i)$, since exterior differentiation is the alternation of covariant differentiation ([K-N] vol. 1, p. 149),

$= 2\theta^*\ d\psi^\gamma(E_j,E_i) = 2\theta^*(\psi^\gamma_\beta \wedge \psi^\beta)(E_j,E_i)$, by the first structural equation.

(b) $A^\gamma_k(\omega^k_i(E_j) - \omega^k_j(E_i)) = \theta^*\psi^\gamma(E_k)\ \omega^k(\nabla_{E_i}E_j - \nabla_{E_j}E_i)$

$= 0$, at the centre of a normal frame.

(c) $A^\alpha_j\ \theta^*\psi^\gamma_\alpha(E_i) - A^\alpha_i \theta^*\psi^\gamma_\alpha(E_j) = \theta^*\psi^\alpha(E_j)\theta^*\psi^\gamma_\alpha(E_i) - \theta^*\psi^\alpha(E_i)\theta^*\psi^\gamma_\alpha(E_j)$

$$= 2\theta^*(\psi^\alpha \wedge \psi^\gamma_\alpha)(E_j,E_i).$$

Thus,

$$
{}^{V}\nabla_{E_i}(d\theta)^{V}(E_j) - {}^{V}\nabla_{E_j}(d\theta)^{V}(E_i)
$$

$$
= 2 \sum_{\beta=1}^{m+q} \sum_{\alpha,\gamma>m} \{\theta^*(\psi^\gamma_\beta \wedge \psi^\beta)(E_j,E_i) - \theta^*(\psi^\gamma_\alpha \wedge \psi^\alpha)\ (E_j,E_i)\}
$$

$$
= 2 \sum_{\gamma>m} \sum_{\beta\le m} \theta^*(\psi^\gamma_\beta \wedge \psi^\beta)(E_j,E_i). \qquad \square
$$

<u>Lemma 3</u>

σ has totally geodesic fibres iff either of the following holds:

(i) $(\nabla_X Y)^H = 0$, whenever X, Y are vertical.

(ii) $(\nabla_X Z)^V = 0$, whenever X is vertical, and Z is horizontal.

<u>Proof</u>. Let $j_x : \sigma^{-1}(x) \to E$ be inclusion. If X, Y are vertical fields, and Z horizontal, then, relative to the isometrically embedded submanifold $\sigma^{-1}(x)$, X and Y are tangential and Z is normal. So, by Weingarten's formula ([K-N] vol. 2, p. 15)

$$
h(\nabla dj_x(X,Y),Z) = h(\nabla_X Y,Z)
$$

from which the first equivalence follows. On the other hand:

$$
h(\nabla dj_x(X,Y),Z) = h(A_Z X,Y) = -h(\nabla_X Z,Y)
$$

where A is the *shape operator* for the fibres. This gives the second equivalence. $\square$

<u>Corollary 1</u>.

σ has totally geodesic fibres iff

$$
\psi^\gamma_\beta(TE)^V = 0, \text{ whenever } \beta \le m \text{ and } \gamma > m.
$$

Proof. Noting the antisymmetry of the connection forms, the condition on the vanishing of ψ^{γ}_{β} is equivalent to both (i) and (ii) in Lemma 3. □

Lemma 4

(TE)^{H} is integrable iff $\psi^{\gamma}_{\beta}(F_{\alpha}) = \psi^{\gamma}_{\alpha}(F_{\beta})$, whenever $\alpha,\beta \le m$ and $\gamma > m$.

Proof.

For horizontal F_{α}, F_{β}:

$$[F_{\alpha},F_{\beta}]^{V} = \sum_{\gamma>m} \psi^{\gamma}(\nabla_{F_{\alpha}}F_{\beta}-\nabla_{F_{\beta}}F_{\alpha})F_{\gamma} = \sum_{\gamma>m} (\psi^{\gamma}_{\beta}(F_{\alpha}) - \psi^{\gamma}_{\alpha}(F_{\beta}))F_{\gamma}.$$

Thus, (TE)^{H} is integrable iff $[F_{\alpha},F_{\beta}]^{V} = 0$

iff $\psi^{\gamma}_{\beta}(F_{\alpha}) = \psi^{\gamma}_{\alpha}(F_{\beta})$, whenever $\alpha,\beta \le m$ and $\gamma > m$. □

When σ is a Riemannian submersion, the preceding results combine to produce the following gloomy outlook:

Proposition 1.

Let σ be a Riemannian submersion. Then:

${}^{V}\nabla(d\theta)^{V}$ is symmetric for every section θ of σ iff σ is totally geodesic.

Proof. By Lemma 2:

${}^{V}\nabla(d\theta)^{V}$ is symmetric, for every section θ

iff $\theta^{*}(\psi^{\gamma}_{\beta} \wedge \psi^{\beta}) = 0$, for every section θ

iff $\psi^{\gamma}_{\beta} \wedge \psi^{\beta} = 0$

iff $\psi^{\gamma}_{\beta} = c^{\gamma}_{\beta}\, \psi^{\beta}$ (no sum), for some $c^{\gamma}_{\beta} : E \to \mathbb{R}$

whenever $\beta \leq m$ and $\gamma > m$.

($\Leftarrow$) Lemma 3.8 of [Vil] states that:

"σ is totally geodesic iff ker $d\sigma$ is holonomy-invariant."

Since parallel translation is isometric, this is equivalent to:

"σ is totally geodesic iff $(\ker d\sigma)^{\perp}$ is holonomy-invariant."

Thus, $\psi^{\gamma}_{\beta} = \psi^{\gamma}\, \nabla F_{\beta} = 0$, whenever $\beta \leq m$ and $\gamma > m$, and the above condition is trivially satisfied by choosing $C^{\gamma}_{\beta} = 0$.

($\Rightarrow$) Theorem 3.3 of [Vil] states that:

"σ is totally geodesic iff σ has totally geodesic fibres, and $(TE)^{H}$ is integrable."

Now, if X is vertical, then $\psi^{\gamma}_{\beta}(X) = C^{\gamma}_{\beta}\, \psi^{\beta}(X) = 0$, whenever $\beta \leq m$ and $\gamma > m$, so that σ has t.g. fibres (by Corollary 1). Also, if α, $\beta \leq m$ and $\gamma > m$, then:

$$\psi^{\gamma}_{\beta}(F_{\alpha}) = C^{\gamma}_{\beta}\psi^{\beta}(F_{\alpha}) = C^{\gamma}_{\beta}\, \delta^{\beta}_{\alpha} = C^{\gamma}_{\alpha}\, \delta^{\alpha}_{\beta} = \psi^{\gamma}_{\alpha}(F_{\beta})$$

so that $(TE)^{H}$ is integrable (by Lemma 4). $\square$

Despite the devastation caused by Proposition 1, we are able to salvage something in the following situation. Let σ have totally geodesic fibres, and θ be a *section* of σ. Suppose that $\Theta: M \times \mathbb{R} \to E$ is a *vertical variation* of θ, in the sense that $\frac{\partial\Theta}{\partial t}\big|_{0}$ is vertical. Then, since geodesics with vertical initial vector remain vertical (because the fibres of σ are totally geodesic), each θ_t is also a section of σ, and we have the factorisation:

$$\begin{tikzcd} & & E \arrow[d, "\sigma"] \\ M \times \mathbb{R} \arrow[urr, "\Theta"] \arrow[rr, "\pi_M"'] & & M \end{tikzcd}$$

Defining $\tilde{\Theta}: M \times \mathbb{R} \to E \times \mathbb{R}$; $\tilde{\Theta}(x,t) = \Theta(x,t),t)$ gives a factorisation of Θ through $E \times \mathbb{R}$:

$$\begin{tikzcd} E \times \mathbb{R} \arrow[r, "\pi_E"] \arrow[d, "\sigma \times id_{\mathbb{R}}"] & E \arrow[d, "\sigma"] \\ M \times \mathbb{R} \arrow[u, bend left, "\tilde{\Theta}"] \arrow[ur, "\Theta"] \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

<u>Lemma 5</u>.

(i) $\sigma \times id_{\mathbb{R}} : (E \times \mathbb{R}, h \times dt^2) \to (M \times \mathbb{R}, g \times dt^2)$ has totally geodesic fibres.

(ii) ${}^V\nabla(d\Theta)^V = d\pi_E \, {}^V\nabla(d\tilde{\Theta})^V$, where $(d\tilde{\Theta})^V$ is the $\sigma \times id_{\mathbb{R}}$-vertical differential of $\tilde{\Theta}$.

<u>Proof</u>.

(i) If $j_{(x,t)} : (\sigma \times id_{\mathbb{R}})^{-1}(x,t) \to E \times \mathbb{R}$ is inclusion, then

$j_{(x,t)} = i_t \circ j_x \circ \pi_E$ is a factorisation with each factor totally geodesic.

(ii) Since $(d\Theta)^V = d\pi_E \circ (d\tilde{\Theta})^V$, we have that:

$${}^V\nabla(d\Theta)^V = \nabla d\pi_E((d\tilde{\Theta})^V, (d\tilde{\Theta})^V) + d\pi_E \, {}^V\nabla(d\tilde{\Theta})^V.$$

Now, $(d\tilde{\Theta})^V$ is a section of $T^*(M \times \mathbb{R}) \otimes \tilde{\Theta}^{-1}(\overline{TE})$, and $\overline{TE}$ is π_E-horizontal so that $\nabla d\pi_E((d\tilde{\Theta})^V, (d\tilde{\Theta})^V) = 0$ (by Corollary 2.5.1 (ii)). □

<u>Proposition 2.</u>

Suppose that σ has totally geodesic fibres, and that Θ is a vertical variation of the *section* θ. Then:

$$^V\nabla(d\Theta)^V(\bar{E}_i, \frac{\partial}{\partial t}) = {}^V\nabla(d\Theta)^V(\frac{\partial}{\partial t}, \bar{E}_i).$$

<u>Proof</u>. Let $\tilde{v} = \frac{\partial\tilde{\Theta}}{\partial t}$. Then:

$$2 \sum_{\gamma>m+1} \sum_{\beta\leq m+1} \tilde{\Theta}^*(\tilde{\psi}^\gamma_\beta \wedge \tilde{\psi}^\beta)(\bar{E}_i, \frac{\partial}{\partial t})$$

$$= \sum_{\gamma>m+1} \sum_{\beta\leq m+1} \{\tilde{\psi}^\gamma_\beta(d\tilde{\Theta} \circ \bar{E}_i)\,\tilde{\psi}^\beta(\tilde{v}) - \tilde{\psi}^\gamma_\beta(\tilde{v})\ \tilde{\psi}^\beta(d\tilde{\Theta} \circ \bar{E}_i)\}.$$

Since $\tilde{v}$ is vertical:

(a) $\tilde{\psi}^\beta(\tilde{v}) = 0$, whenever $\beta \leq m+1$.

(b) $\tilde{\psi}^\gamma_\beta(\tilde{v}) = 0$, whenever $\beta \leq m+1$ and $\gamma > m+1$ (by Lemmas 3 and 5(i)).

Thus, by Lemma 2:

$$^V\nabla(d\tilde{\Theta})^V(\bar{E}_i, \frac{\partial}{\partial t}) = {}^V\nabla(d\tilde{\Theta})^V(\frac{\partial}{\partial t}, \bar{E}_i)$$

and the result follows from Lemma 5 (ii). □

We can now press ahead with the calculation of the Euler-Lagrange equations for the vertical energy functionals. The analogue of Proposition 1.3.3 is:

Lemma 6.

Let σ have totally geodesic fibres, and let Θ be a vertical variation of a section θ of σ, with variation field v. Then $\tilde{\Theta}^*h^V$ has variation field:

$$\left.\frac{d\theta_t^*h^V}{dt}\right|_s = 2\ \mathrm{Sym}\ \{h^V(d\theta_s(E_i),\nabla_{E_j}v_s)\ \omega^i \otimes \omega^j\}.$$

Proof.

In analogy with the proof of Proposition 1.3.3:

$$\left.\frac{d\theta_t^*h^V}{dt}\right|_s = \frac{\partial}{\partial t}\ \Theta^*h^V(\bar{E}_i,\bar{E}_j) \circ i_s, \text{ and:}$$

$$\frac{\partial}{\partial t}\ \Theta^*h^V(\bar{E}_i,\bar{E}_j) = \frac{\partial}{\partial t}\ h((d\Theta)^V \circ \bar{E}_i,\ (d\Theta)^V \circ \bar{E}_j)$$

$$= h(\nabla_{\frac{\partial}{\partial t}}[(d\Theta)^V\circ\bar{E}_i],(d\Theta)^V\circ\bar{E}_j) + h((d\Theta)^V\circ\bar{E}_i,\nabla_{\frac{\partial}{\partial t}}[(d\Theta)^V\circ\bar{E}_j])$$

$$= h^V(\nabla_{\bar{E}_i}v\ ,d\Theta(\bar{E}_j)) + h^V(d\Theta(\bar{E}_i),\nabla_{\bar{E}_j}v), \text{ by Proposition 2.}$$

Thus,

$$\frac{\partial}{\partial t}\ \Theta^*h^V(\bar{E}_i,\bar{E}_j) \circ i_s = h^V(\nabla_{E_i}v_s,d\theta_s(E_j))+h^V(d\theta_s(E_i),\nabla_{E_j}v_s).$$

□

When Lemma 6 is substituted for Proposition 1.3.3, the following have proofs identical to those of Lemmas 1.6.2 and 1.6.3:

Lemma 7.

With the same hypotheses as Lemma 6:

$$\left.\frac{d}{dt}\right|_s \sigma_r^V(\theta_t) = 2 <(d\theta_s)^V \circ \chi_{r-1}^V(\theta_s),\ \nabla v_s> . \qquad \square$$

Lemma 8.

With the same hypotheses as Lemma 6:

$$\frac{d}{dt}\Big|_s E_r^V(\theta_t) = -2 \int_M h^V(\text{Trace} \quad \nabla((d\theta_s)^V \circ \chi_{r-1}^V(\theta_s)), v_s) v_g. \quad \square$$

Theorem.

Let $\sigma:(E,h) \to (M,g)$ have *totally geodesic fibres*, and let θ be a *section* of σ. Then θ is a critical point of E_r^V w.r.t. *variations through sections*

$$\text{iff } \tau_r^V(\theta,g) = 0. \quad \square$$

Remark 1. If in addition σ is a Riemannian submersion and E is complete, then $\sigma:E \to M$ has isometric fibres and may be given the structure of a (G,F)-bundle with connection, where F denotes the isometry-type of the fibres, and G the isometry group of F ([Her]). Thus, from the variational viewpoint there is little loss of generality in considering sections of a fibre bundle with connection.

Remark 2. Let $\phi:(M,g) \to (N,h)$, $\sigma = \pi_M:(M \times N,\ g \times h) \to (M,g)$ and $\theta = \Gamma_\phi$. Then, σ has totally geodesic fibres, θ is a section of σ, $E_r^V(\theta) = E_r(\phi)$, and the vertical variations of θ correspond to arbitrary variations of ϕ. So:

ϕ is a critical point of E_r

iff θ is a critical point of E_r^V w.r.t. vertical variations

iff Γ_ϕ is a r-harmonic section of π_M, by the Theorem,

thus verifying Theorem 1.1.

§3. HARMONIC SECTIONS OF A RIEMANNIAN VECTOR BUNDLE

Let $\sigma : E \to (M,g)$ be a vector bundle with connection map K (cf. Chapter 2, §6) and fibre metric $\langle , \rangle$; we do not at this stage assume compatibility of $\langle , \rangle$ with K. By horizontally lifting g, and vertically lifting $\langle , \rangle$, we give E the Riemannian metric h:

$$h(v,w) = \langle Kv, Kw \rangle + g(d\sigma(v), d\sigma(w)).$$

Hopeful of avoiding ambiguity, we shall use ∇ to denote the covariant derivative for both K and the Riemannian connection of (E,h).

If $X \in \mathfrak{C}(TM)$ and $\alpha \in \mathfrak{C}(E)$, let $\tilde{X}$ denote the horizontal lift of X, and $\tilde{\alpha}$ the *vertical lift* of α, characterized by:

$$K \circ \tilde{\alpha} = \alpha \circ \sigma.$$

Let ξ_t be the flow of X, and $\tilde{\xi}_t$ that of $\tilde{X}$.

Remark 1. $\tilde{\xi}_t$ is the horizontal lift of ξ_t; thus, $\tilde{\xi}_t = \sigma_t^{\xi}$.

Lemma 1.

(i) $d\xi_t$ preserves $(TE)^V$, and the following diagram commutes:

$$\begin{tikzcd} (TE)^V \arrow[r, "d\tilde{\xi}_t"] \arrow[d, "K"'] & (TE)^V \arrow[d, "K"] \\ E \arrow[r, "\tilde{\xi}_t"'] & E \end{tikzcd}$$

(ii) If $d\tilde{\xi}_t$ preserves $(TE)^H$, then $(TE)^H$ is integrable.

Proof.

(i) For any $x \in M$, the restriction $\tilde{\xi}_t : E_x \to E_{\xi_t(x)}$ is linear, by the Remark. Then, recalling Lemma 2.6.1, each $e \in E_x$ gives the following commutative diagram:

$$\begin{tikzcd}
(T_e E)^V \arrow[r, "d\tilde{\xi}_t"] & (T_{\tilde{\xi}_t(e)} E)^V \\
E_x \arrow[u, "j_e"] \arrow[r, "\tilde{\xi}_t"'] & E_{\xi_t(x)} \arrow[u, "j_{\tilde{\xi}_t(e)}"']
\end{tikzcd}$$

But, by Proposition 2.6.1, K is a left inverse for all the maps j_e.

(ii) $d\tilde{\xi}_t$ preserves $(TE)^V$ <==> parallel translation in σ preserves horizontality of curves

$<\!\!==>$ $(\nabla_Y Z)^V = 0$, for all horizontal Y,Z.

$==>$ $[Y,Z]^V = 0$, for all horizontal Y,Z $<\!\!==>$ $(TE)^H$ is integrable. □

Lemma 2.

$$K\,[\tilde{X},\tilde{\alpha}] = \nabla_X\alpha$$

Proof.

$$K[\tilde{X},\tilde{\alpha}] = K\left.\frac{d}{dt}\right|_0 d\tilde{\xi}_{-t}\circ\tilde{\alpha}\circ\tilde{\xi}_t = \left.\frac{d}{dt}\right|_0 \tilde{\xi}_{-t}\circ K\circ\tilde{\alpha}\circ\tilde{\xi}_t, \text{ by Lemma 1}$$

$$= \left.\frac{d}{dt}\right|_0 \tilde{\xi}_{-t}\circ\alpha\circ\xi_t = \left.\frac{d}{dt}\right|_0 (\sigma^{\xi}{}_t)^{-1}\circ\alpha\circ\xi_t, \text{ by the Remark}$$

$$= \nabla_X\alpha. \quad \square$$

Lemma 3.

$\sigma:(E,h) \to (M,g)$ has totally geodesic fibres iff $(E,<,>,K)$ is a Riemannian vector bundle.

Proof. By Lemma 2.3, it suffices to show that $h(\nabla_{\tilde{\alpha}}\tilde{X},\tilde{\beta}) = 0$, for every α, $\beta \in \mathbf{C}(E)$ and $X \in \mathbf{C}(TM)$. The Riemannian connection for (E,h) is characterised by ([K-N]) vol. 1, p. 160):

$$2h(\nabla_{\tilde{\alpha}}\tilde{X},\tilde{\beta}) = \tilde{\alpha}.h(\tilde{X},\tilde{\beta}) + \tilde{X}.h(\tilde{\beta},\tilde{\alpha}) - \tilde{\beta}.h(\tilde{\alpha},\tilde{X})$$

$$+ h([\tilde{\alpha},\tilde{X}],\tilde{\beta}) + h([\tilde{\beta},\tilde{\alpha}],\tilde{X}) - h([\tilde{X},\tilde{\beta}],\tilde{\alpha})$$

$$= \tilde{X}.h(\tilde{\beta},\tilde{\alpha}) - h([\tilde{X},\tilde{\alpha}],\tilde{\beta}) - h([\tilde{X},\tilde{\beta}],\tilde{\alpha})$$

$$= X.<\beta,\alpha> - <\nabla_X\alpha,\beta> - <\nabla_X\beta,\alpha>, \text{ by Lemma 2}$$

$$= 0 \text{ iff } <,> \text{ and } \nabla \text{ are compatible.} \qquad \square$$

Lemma 4.

If $(E,<,>,K)$ is a Riemannian vector bundle, and α a section of E, then:

$$K\ {}^V\nabla(d\alpha)^V = \nabla^2\alpha$$

Proof. For $X,Y \in \mathbf{C}(TM)$:

$$\nabla_X(d\alpha)^V(Y) = \nabla_X((d\alpha)^V \circ Y) - (d\alpha)^V\nabla_X Y.$$

To evaluate $\nabla_X((d\alpha)^V \circ Y)$, we note that $(d\alpha)^V \circ Y = (\nabla_Y\alpha)^{\sim} \circ \alpha$ (by Corollary 2.6.1), and $(d\alpha)^H \circ Y = \tilde{Y} \circ \alpha$. Let us therefore extend $d\alpha(Y)$ away from the submanifold $\alpha(M)$ to a vector field Y^α on E by defining:

$$(Y^\alpha)^V = (\nabla_Y \alpha)^\sim, \quad (Y^\alpha)^H = \tilde{Y}.$$

Then, $d\alpha \circ Y = \alpha^{-1}(Y^\alpha)$ so that, recalling Chapter 1 §2:

$$\nabla_X((d\alpha)^V \circ Y) = \nabla_{X^\alpha}(Y^\alpha)^V = \nabla_{(Y^\alpha)^V} X^\alpha + [X^\alpha, (Y^\alpha)^V].$$

On taking vertical components:

$$^V\nabla_X((d\alpha)^V \circ Y) = \nabla_{(Y^\alpha)^V}(X^\alpha)^V + [(X^\alpha)^V, (Y^\alpha)^V] + [(X^\alpha)^H, (Y^\alpha)^V],$$

by Lemmas 3 and 2.3

$$= \nabla_{(X^\alpha)^V}(Y^\alpha)^V + [\tilde{X}, (\nabla_Y \alpha)^\sim]$$

$$= \nabla_{(X^\alpha)^V}(Y^\alpha)^V + (\nabla_X \nabla_Y \alpha)^\sim, \text{ by Lemma 2.}$$

Since each fibre of σ is a flat, totally geodesic vector space, in which the trajectories of vertical lifts of sections of E are just parallel lines, we have that:

$$\nabla_{(X^\alpha)^V}(Y^\alpha)^V = 0.$$

Thus:

$$K\, {}^V\nabla_X (d\alpha)^V(Y) = \nabla_X \nabla_Y \alpha - \nabla\alpha(\nabla_X Y) = \nabla^2\alpha(Y,X). \quad \Box$$

<u>Proposition 1</u>.

α is a harmonic section of a Riemannian vector bundle iff

$$\text{Trace } \nabla^2\alpha = 0. \quad \Box$$

Remark 2. If α is an E-valued p-form, then - Trace $\nabla^2\alpha$ is the *rough Laplacian* of α. Thus, the harmonic sections of $\Lambda^p T^*M \otimes E$ are precisely the "roughly harmonic" E-valued p-forms.

We now consider the variational aspect of compactly supported harmonic sections of σ. In view of Lemma 3 and Theorem 2.1, it is natural to insist that (E,<,>,K) be a Riemannian vector bundle.

Proposition 2. (cf. [Nou])

If (E,<,>K) is a Riemannian vector bundle with compact base, then:

$$\alpha \text{ is a harmonic section iff } \nabla\alpha = 0.$$

Proof. $2e^V(\alpha) = \|(d\alpha)^V\|^2 = \sum_i \|(d\alpha)^V(E_i)\|^2$, for some orthonormal frame $\{E_i\}$.

$$= \sum_i \langle\nabla_{E_i}\alpha,\nabla_{E_i}\alpha\rangle \text{ (by Corollary 2.6.1) } = \|\nabla\alpha\|^2.$$

We recall that if (V,<,>) is any Hilbert space, and $Q(x) = \|x\|^2$, then $dQ(x)(y) = 2\langle x,y\rangle$, for all $x,y \in V$. Thus, writing <<,>> for the L^2 inner product on $\mathfrak{C}(T^*M \otimes E)$, we have that $E^V(\alpha) = \frac{1}{2}\langle\langle\nabla_\alpha,\nabla_\alpha\rangle\rangle$ and $dE^V(\alpha)(\tilde{\beta}) = \langle\langle\nabla_\alpha,\nabla_\beta\rangle\rangle$ for any $\beta \in \mathfrak{C}(E)$ (with vertical lift $\tilde{\beta}$).
(This follows from the chain rule, bearing in mind that the map $\mathfrak{C}(E) \to \mathfrak{C}(T^*M \otimes E)$; $\beta \to \nabla\beta$ is $\mathbb{R}$-linear.) Hence, for any vertical variation of α, with variation field v:

$$dE^V(\alpha)(v) = \langle\langle\nabla\alpha, \nabla(Kv)\rangle\rangle.$$

By choosing a variation with $Kv = \alpha$, the result follows from Theorem 2.1. $\square$

Remark 3. The compactness assumption may be dropped by insisting that α have *finite vertical energy* i.e. $\int_M \|\nabla\alpha\|^2 v_g < \infty$.

Corollary

$\phi:(M,g) \to (N,h)$ is a harmonic map iff $\tau(\phi)$ is a harmonic section of $\phi^{-1}(TN)$.

Proof. ϕ is harmonic iff $d\phi$ is a harmonic $\phi^{-1}(TN)$-valued 1-form (cf. [E-L], 2-15).

$$\text{iff } 0 = \Delta d\phi = -\nabla\tau(\phi). \quad \square$$

Combining Propositions 1 and 2, we have the following reduction theorem:

Theorem.

If $(E,\langle\,,\rangle,K)$ is a Riemannian vector bundle, and α a section of finite vertical energy, then

$$\text{Trace } \nabla^2\alpha = 0 \quad \text{iff } \nabla\alpha = 0. \quad \square$$

Remark 4. By considering the product bundle $M \times \mathbb{R}^n \to M$ with trivial connection, and applying Theorem 1.1, we recover the fact that every harmonic map $(M,g) \to \mathbb{R}^n$ of finite energy is constant.

§4. EXISTENCE AND UNIQUE CONTINUATION FOR HARMONIC SECTIONS

We recall the basic existence and unique continuation properties of harmonic maps:

Theorem 1. [Sam]

If two harmonic maps $(M,g) \to (N,h)$ agree in a (non-empty) open set, then they agree everywhere. □

Theorem 2. [E-S]

If M,N are compact, and (N,h) has non-positive sectional curvature, then any C^1 map $(M,g) \to (N,h)$ can be smoothly deformed into a harmonic map. □

Let $\sigma:(E,h) \to (M,g)$ be a (G,Q)-bundle associated to $\xi:(P,k) \to (M,g)$. To state analogous results for harmonic sections of σ, we exploit Theorem 1.2.

Theorem 3.

If two harmonic sections of σ agree on a (non-empty) open set, they agree everywhere.

Proof. Let $\theta_1,\theta_2:M \to E$ be sections, and $\phi_1,\phi_2:P \to Q$ the corresponding equivariant maps. If $\theta_1(x) = \theta_2(x)$ for some $x \in M$, then:

$$\phi_1(p) = \mu_p^{-1}\circ\theta_1\circ\xi(p) = \mu_p^{-1}\circ\theta_2\circ\xi(p) = \phi_2(p), \text{ for every } p \in \xi^{-1}(x)$$

Thus, if $U \subset M$ is open and $\theta_1(x) = \theta_2(x)$ for all $x \in U$, then $\phi_1(p) = \phi_2(p)$ for all $p \in \xi^{-1}(U)$, and it follows from Theorem 1 that $\phi_1 = \phi_2$. Hence $\theta_1 = \theta_2$. □

The Eells-Sampson proof of Theorem 2 is based on deformation along the trajectories of the heat equation. It goes roughly as follows:

(1) A solution to the heat equation on $[0,t)$ with initial value $\phi \in C^1(P,Q)$ is a map $\Phi: P \times [0,t) \to Q$ with $\phi \in C^1(P \times [0,t), Q) \cap C^\infty(P \times (0,t), Q)$ and satisfying:

$$\mathcal{L}(\Phi)(x,t) = \tau(\phi_t)(x) - \frac{\partial \Phi}{\partial t}(x,t) = 0, \text{ and } \phi_0 = \phi.$$

(2) When (Q,ℓ) has non-positive Riemannian sectional curvature, a unique solution to the heat equation exists on $[0,\infty)$, for any C^1 initial condition.

(3) If in addition Q is compact, then $\phi_\infty(x) = \lim_{t\to\infty} \phi_t(x)$ exists as a C^k limit for all $k > 0$, and ϕ_∞ is a harmonic map.

If P and Q are (left) G-manifolds, then so is $C^1(P,Q)$:

$$g.\phi(x) = g\phi(g^{-1}x), \text{ for all } g \in G.$$

By giving $\mathbb{R}$ the trivial G-action, $C^1(P \times \mathbb{R}, Q)$ is also a G-manifold.

<u>Remark 1.</u> ϕ is G-equivariant iff ϕ is a fixed point of G.

<u>Remark 2.</u> $\Phi \in C^1(P \times \mathbb{R}, Q)$ is G-equivariant iff each ϕ_t is G-equivariant. For:

$$g.\Phi(x,t) = g\Phi(g^{-1}(x,t)) = g\Phi(g^{-1}x,t) = g\phi_t(g^{-1}x) = g.\phi_t(x).$$

Thus,

$$G \text{ fixes } \Phi \text{ iff } G \text{ fixes each } \phi_t$$

and the result follows from Remark 1.

Say that (P,k), (Q,ℓ) are *Riemannian G-manifolds* (i.e. G acts on P,Q by isometries). Let L_g denote the isometry of P,Q or $P \times \mathbb{R}$ corresponding to $g \in G$.

Remark 3. By Corollary 2.5.1 (i), each L_g is totally geodesic.

To prove the analogue of Theorem 2 for harmonic sections, it suffices to show that the heat flow between two Riemannian G-manifolds preserves equivariance.

Lemma

$$(g.\Phi)(gx,t) = dL_g\, \mathcal{L}(\Phi)(x,t).$$

Proof.

$$\mathcal{L}(g.\Phi)(gx,t) = \tau(g.\phi_t)(gx) - \frac{\partial(g.\phi)}{\partial t}(gx,t).$$

Now,

$$\frac{\partial(g.\phi)}{\partial t}(gx,t) = d(L_g \circ \Phi \circ L_g^{-1})(\frac{\partial}{\partial t}|_{(gx,t)})$$

$$= dL_g \circ d\Phi \circ dL_g^{-1} \circ dL_g(\frac{\partial}{\partial t}|_{(x,t)}) = dL_g \frac{\partial\Phi}{\partial t}(x,t).$$

If $\{E_i\}$ is an orthonormal frame at x, then:

$$\tau(g.\phi_t)(gx) = \sum_i \nabla d(L_g \circ \phi_t \circ L_g^{-1})(dL_g(E_i),dL_g(E_i))$$

$$= \sum_i \{\nabla dL_g(d\phi_t(E_i),d\phi_t(E_i)) + dL_g\, \nabla d\phi_t(E_i,E_i)$$

$$+ d(L_g \circ \phi_t)\, \nabla dL_g^{-1}(dL_g(E_i),dL_g(E_i))\}$$

$$= \sum_i dL_g\, \nabla d\phi_t(E_i,E_i), \text{ by Remark 3.}$$

$$= dL_g\, \tau(\phi_t)(x).$$

Thus,

$$\mathcal{L}(g.\Phi)(gx,t) = dL_g\{\tau(\phi_t)(x) - \frac{\partial\Phi}{\partial t}(x,t)\} = dL_g\, \mathcal{L}(\Phi)(x,t). \quad \square$$

Theorem 4.

Let M be compact. If (Q,ℓ) is a compact Riemannian G-manifold with non-positive sectional curvature, then any C' section of $(E = P\times_G Q,h) \to (M,g)$ can be smoothly deformed through sections to a harmonic section.

Proof. Let G' be the isometry group of (Q,ℓ), and $\xi':P' \to M$ the principal G'-bundle associated to E. If $\lambda:G \to G'$ is the action of G on Q, then ξ' is a λ-extension of ξ. Since Q is compact, so is G' ([K-N] vol. 1, p. 239) and hence P'.

Let θ be a C' section of E, with corresponding C' G'-equivariant map $\phi':P' \to Q$; then, by Theorem 2, ϕ' can be deformed by heat flow Φ' to a harmonic map ϕ'_∞. By the Lemma, $g'\Phi'$ is the unique solution of the heat equation with initial condition $g'\phi'$, for any $g' \in G'$. But, since ϕ' is equivariant, the initial conditions all coincide (Remark 1) so that $g'\Phi' = \Phi'$ for all $g' \in G'$. Thus, Φ' is equivariant (Remark 1), and hence all ϕ'_t (Remark 2), giving an equivariant harmonic limit ϕ'_∞. Consequently, there is a corresponding deformation θ_t of θ through sections, with the limiting section θ_∞ harmonic (Theorem 1.2). $\square$

Remark 4. The compactness of Q may be weakened to certain growth conditions ([E-S]), and Theorem 4 carries over in such cases where the isometry group remains compact.

CHAPTER 4 : THE GAUSS SECTION

§1. THE RUH-VILMS THEOREM

Suppose that $\phi:(M^m,g) \to \mathbb{R}^n$ is a *Riemannian immersion* (where $\mathbb{R}^n$ is given the Euclidean metric), and let $\gamma_\phi : M \to G_{m,n}$ be its *Gauss map* into the Grassmannian of m-planes in $\mathbb{R}^n$:

$$\gamma_\phi(x) = \{v \in \mathbb{R}^n : (\phi(x),v) \in d\phi(T_xM)\}$$

where $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. Let H_ϕ denote the *mean curvature normal* field of ϕ, and D be the connection in the normal bundle of ϕ. The geometry of ϕ is then related to that of γ_ϕ by the *Ruh-Vilms Theorem:*

Theorem [R-V]

γ_ϕ is harmonic (w.r.t. the usual Riemannian metric on $G_{m,n}$)

iff $DH_\phi = 0$.

Proof.

We provide six steps, the technical details of which are deferred to a subsequent series of Propositions.

Firstly, we recall that the *canonical* and *ortho-complement* bundles of $G_{m,n}$ are the following vector sub-bundles of $G_{m,n} \times \mathbb{R}^n \to G_{m,n}$:

$$K_{m,n} = \{(V,v) \in G_{m,n} \times \mathbb{R}^n : v \in V\}; \quad K^\perp_{m,n} = \{(V,v) \in G_{m,n} \times \mathbb{R}^n : v \perp V\}.$$

Then, $G_{m,n} \times \mathbb{R}^n = K_{m,n} \oplus K^\perp_{m,n}$, and $K_{m,n}$, $K^\perp_{m,n}$ each inherit their geometry from $G_{m,n} \times \mathbb{R}^n$ via the projection morphisms

$p: G_{m,n} \times \mathbb{R}^n \to K_{m,n}$ and $p^{\perp}: G_{m,n} \times \mathbb{R}^n \to K^{\perp}_{m,n}$; for example, $(TK_{m,n})^H_{(V,v)} = dp(\overline{TG}_{m,n})_{(V,v)}$.

Remark 1. The use of the projections is necessary because the sub-bundles $K_{m,n}$ and $K^{\perp}_{m,n}$ are not stable under the (trivial) parallel translation of $G_{m,n} \times \mathbb{R}^n$ i.e. the natural connection in $G_{m,n} \times \mathbb{R}^n$ does not reduce to $K_{m,n}$ and $K^{\perp}_{m,n}$.

Remark 2. Covariant differentiation in $K_{m,n}$ (or $K^{\perp}_{m,n}$) is the projection of covariant differentiation in $G_{m,n} \times \mathbb{R}^n$, and the latter is just ordinary differentiation:

$$\nabla_X \Gamma_v(V) = (V, dv(V)(X))$$

where $V \in G_{m,n}$, $X \in T_V G_{m,n}$, $v: G_{m,n} \to \mathbb{R}^n$, and Γ_v = graph v (representing a typical section of $G_{m,n} \times \mathbb{R}^n \to G_{m,n}$).

(1) There is a natural vector bundle morphism A:

$$\begin{array}{ccc} \mathrm{Hom}(TM, TM^{\perp}) & \xrightarrow{A} & \mathrm{Hom}(K_{m,n}, K^{\perp}_{m,n}) \\ \downarrow & & \downarrow \\ M & \xrightarrow[\gamma_\phi]{} & G_{m,n} \end{array}$$

which maps fibres isomorphically (i.e. is an isomorphism to the pullback bundle), and is connection-preserving (Proposition 1).

(2) There is a natural connection-preserving vector bundle isometry B (Proposition 3):

$$\begin{tikzcd}
TG_{m,n} \arrow[rr, "B"] \arrow[dr] & & \mathrm{Hom}(K_{m,n}, K^{\perp}_{m,n}) \arrow[dl] \\
& G_{m,n} &
\end{tikzcd}$$

(3) The composition $C = B^{-1} \circ A$ is thus a connection-preserving fibre isomorphism:

$$\begin{tikzcd}
\mathrm{Hom}\,(TM, TM^{\perp}) \arrow[r, "C"] \arrow[d] & TG_{m,n} \arrow[d] \\
M \arrow[r, dash, "\gamma_\phi"'] & G_{m,n}
\end{tikzcd}$$

(4) The fundamental form of $d\phi$ is related to the differential of γ_ϕ (Proposition 4):

$$C(\nabla_X d\phi) = d\gamma_\phi(X), \text{ for any } X \in TM.$$

(5) Since ϕ immerses (M,g) isometrically into a space of constant sectional curvature, Codazzi's equation for ϕ reduces to ([K-N] vol. 2, p. 25):

$$D_X(\nabla d\phi)(Y,Z) = D_Y(\nabla d\phi)(X,Z), \text{ for all } X,Y,Z \in TM.$$

(6) (a) $\nabla_X(d\gamma_\phi \circ Y) = \nabla_X d\gamma_\phi(Y) + d\gamma_\phi(\nabla_X Y)$

(b) Thinking of $\nabla d\phi$ as a section of $\mathrm{Hom}\,(TM, \mathrm{Hom}(TM, TM^{\perp}))$:

$$\nabla_X(C(\nabla_Y d\phi)) \underset{(3)}{=} C\{D_X(\nabla d\phi(Y))\} = C\{(D_X(\nabla d\phi)(Y) + \nabla d\phi(\nabla_X Y)\}$$

$$\underset{(4)}{=} C\{D_X(\nabla d\phi)(Y)\} + d\gamma_\phi(\nabla_X Y) \underset{(5)}{=} C\{D(\nabla d\phi)(X,Y)\} + d\gamma_\phi(\nabla_X Y)$$

$$= C\{D(\nabla d\phi(X,Y)) - \nabla d\phi(\nabla X,Y) - \nabla d\phi(X,\nabla Y)\} + d\gamma_\phi(\nabla_X Y).$$

Using (4) to equate (a) with (b):

$$\nabla d\gamma_\phi(X,Y) = C\{D(\nabla d\phi(X,Y)) - \nabla d\phi(\nabla X,Y) - \nabla d\phi(X,\nabla Y)\}.$$

We note that ([K-N] vol. 2, p. 34; Corollary 2.5.1 (i)):

$$H_\phi = \frac{1}{m}\,\text{Trace}\ (\nabla d\phi).$$

Thus, if $\{E_i\}_1^m$ is a local frame field in M which is Riemannian normal at $x \in M$ (i.e. $\nabla E_i(x) = 0$, $g(E_i,E_j) = \delta_{ij}$):

$$\tau(\gamma_\phi) = \sum_i \nabla d\gamma_\phi(E_i,E_i) = C\{D(\sum_i \nabla d\phi(E_i,E_i))$$

$$= C\{D\ \text{Trace}\ \nabla d\phi\} = mC\ (DH_\phi).$$

Thus, since C is a fibre isomorphism, $\tau(\gamma_\phi) = 0$ iff $DH_\phi = 0$. □

Remark 3. In general, for a Riemannian immersion $\phi:(M,g) \to (N,h)$:

$$\nabla_X \|H_\phi\|^2 = 2h(\nabla_X H_\phi, H_\phi) = 2h(D_X H_\phi, H_\phi)$$

so that $DH_\phi = 0 \implies \|H_\phi\| = \text{const.}$; the converse is true when $\phi(M)$ is a hypersurface. In the latter case, the mean curvature is just a $\mathbb{R}$-valued function (unique up to choice of sign) which is thereby constant whenever $DH_\phi = 0$. We have:

Corollary

If $\phi(M)$ is a hypersurface, then γ_ϕ is harmonic iff ϕ has constant mean curvature. □

We recall that $\phi^{-1}(T\mathbb{R}^n) = \{(x,(\phi(x),v)) : x \in M,\ v \in \mathbb{R}^n\}$, when $T\mathbb{R}^n$ is identified with $\mathbb{R}^n \times \mathbb{R}^n$. Define a vector bundle morphism I:

$$\begin{tikzcd} \phi^{-1}(T\mathbb{R}^n) \arrow[r, "I"] \arrow[d] & G_{m,n} \times \mathbb{R}^n \arrow[d] \\ M \arrow[r, "\gamma_\phi"'] & G_{m,n} \end{tikzcd} \quad ; \; I(x,(\phi(x),v)) = (\gamma_\phi(x)v)$$

Restricting I to TM (resp. $TM^\perp$) gives a morphism I_1 (resp. I_2):

$$\begin{tikzcd} TM \arrow[r, "I_1"] \arrow[d] & K_{m,n} \arrow[d] \\ M \arrow[r, dash, "\gamma_\phi"'] & G_{m,n} \end{tikzcd} \qquad \begin{tikzcd} TM^\perp \arrow[r, "I_2"] \arrow[d] & K^\perp_{m,n} \arrow[d] \\ M \arrow[r, dash, "\gamma_\phi"'] & G_{m,n} \end{tikzcd}$$

Conjugating by I gives a morphism A:

$$\begin{tikzcd} \mathrm{End}\ \phi^{-1}(T\mathbb{R}^n) \arrow[r, "A"] \arrow[d] & \mathrm{End}\ (G_{m,n} \times \mathbb{R}^n) \arrow[d] \\ M \arrow[r, "\gamma_\phi"'] & G_{m,n} \end{tikzcd}$$

which when restricted to Hom $(TM,TM^\perp)$ maps into Hom $(K_{m,n},K^\perp_{m,n})$; $A = I_2 \circ \mathcal{L} \circ I_1^{-1}$, for any $\mathcal{L} \in$ Hom $(TM,TM^\perp)$. All these morphisms are fibre isomorphisms. Moreover:

<u>Proposition 1</u>.

A is connection-preserving.

<u>Proof</u>. It suffices to show that I is *horizontal* (i.e. dI preserves horizontality of tangent vectors), from which follows the horizontality of I_1, I_2 and hence A. The horizontal distribution for $G_{m,n} \times \mathbb{R}^n$ is $\overline{TG}_{m,n}$, and that of $\phi^{-1}(T\mathbb{R}^n)$ is characterised as the unique sub-bundle rendering the pullback morphism

$$\begin{array}{ccc} \phi^{-1}(T\mathbb{R}^n) & \xrightarrow{\tilde{\phi}} & T\mathbb{R}^n \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & \mathbb{R}^n \end{array} \quad \text{horizontal.}$$

Let $c(t)$ be a curve in $\phi^{-1}(T\mathbb{R}^n)$, generating paths $x(t)$ in M and $v(t)$ in $\mathbb{R}^n$ such that $c(t) = (x(t), (\phi \circ x(t), v(t)))$. Then:

$c'(0)$ is horizontal iff $d\tilde{\phi}(c'(0))$ is horizontal in $TT\mathbb{R}^n$

iff $\frac{d}{dt}\big|_0 (\phi \circ x(t), v(t))$ is horizontal iff $v'(0) = 0$.

$dI(c'(0))$ is horizontal iff $\frac{d}{dt}\big|_0 (\gamma_\phi \circ x(t), v(t))$ is horizontal in $T(G_{m,n} \times \mathbb{R}^n)$

iff $v'(0) = 0$.

Thus, $c'(0)$ is horizontal iff $dI(c'(0))$ is horizontal. $\square$

The isomorphism B may be described informally as follows (see Figure 3, and cf. [M-S] Chapter 5):-

Let $V(t)$ be a path in $G_{m,n}$, and let $v_0 \in V(0)$, so that $(V_0, v_0) \in (K_{m,n})_{V_0}$. For each t there is a unique *orthogonal rotation* of $V(0)$ onto $V(t)$ (i.e. a rotation whose initial velocity field at $V(0)$ is orthogonal to $V(0)$), which carries v_0 to $v(t)$, say. We thereby have a linear map:

$$(K_{m,n})_{V(0)} \to (K_{m,n})_{V(t)}; \ (V_0, v_0) \to (V(t), v(t)).$$

Orthogonal projection of $V(t)$ onto $V(0)^\perp$ (along $V(0)$) produces a linear map:

$$(K_{m,n})_{V(t)} \to (K^\perp_{m,n})_{V(0)}; \ (V(t), v(t)) \to (V(0), v(t)^\perp).$$

Composition then gives us the curve of linear maps:

$$\mathcal{L}(V_t):(K_{m,n})_{V(0)} \to (K^{\perp}_{m,n})_{V(0)};\ (V_o,v_o) \to (V(0),v(t)^{\perp}),$$

and we define $B:TG_{m,n} \to \mathrm{Hom}(K_{m,n},K^{\perp}_{m,n});\ \frac{d}{dt}\big|_o V(t) \to \frac{d}{dt}\big|_o \mathcal{L}(V_t)$.

Then, $B(V'(0))(V_o,v_o) = (V_o, \frac{d}{dt}\big|_o v(t)^{\perp})$.

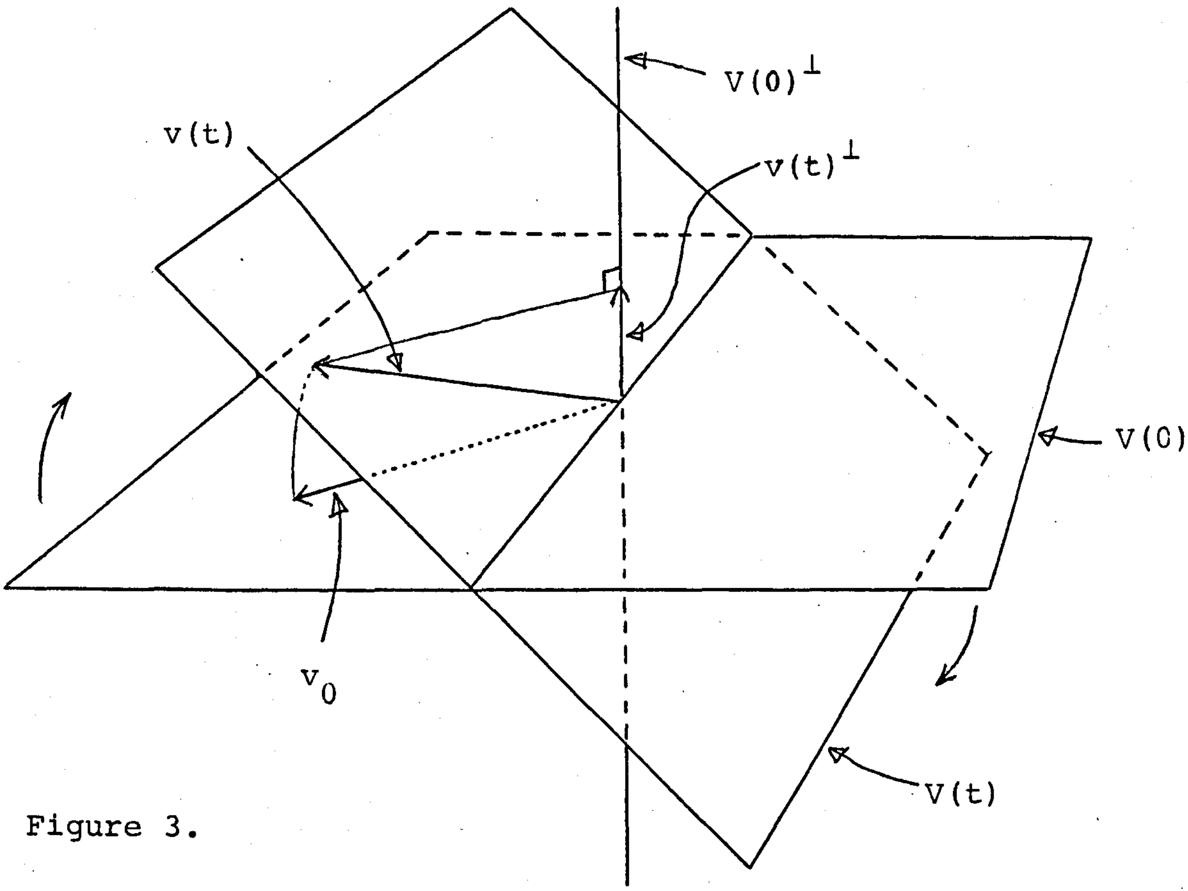

Figure 3.

The geometrical calculations of Propositions 3 and 4 require a more precise definition of B, based on the realisation of $G_{m,n}$ as a symmetric homogeneous space ([K-N] vol 2, p. 271):

The usual left action of $O(n)$ on $\mathbb{R}^n$ gives a transitive left $O(n)$-action on $G_{m,n}$, namely, $P.V = \{P.v : v \in V\}$, for any $P \in O(n)$, $V \in G_{m,n}$. Fixing the point of action to be the "standard" m-plane $V_{can} = \mathrm{span}\{e_1,\ldots,e_m\}$ (where $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbb{R}^n$), the equivariant map $\pi:O(n) \to G_{m,n}$ thus defined factors through the quotient of $O(n)$ by the V_{can}-isotropy subgroup to give a bijection:

$$\begin{tikzcd} O(n) \arrow[r, "\pi"] \arrow[d] & G_{m,n} \arrow[dl, leftrightarrow] \\ O(n)/O(m) \times O(n-m) & \end{tikzcd}$$

The Riemannian metric on $G_{m,n}$ is then that unique metric w.r.t. which π is a Riemannian submersion. Since π is equivariant, and the natural metric on $O(n)$ is invariant, the complementary distributions $\ker d\pi$ and $(\ker d\pi)^{\perp}$ are determined by their "values" at a single point; put $\mathfrak{h} = \ker d\pi(\mathbb{1})$ $(= \mathfrak{o}(m) \times \mathfrak{o}(n-m))$, and $\mathfrak{m} = \mathfrak{h}^{\perp}$.

The product $O(n)$-action on $G_{m,n} \times \mathbb{R}^n$ restricts to give (non-fibre-preserving) actions on $K_{m,n}$ and $K^{\perp}_{m,n}$, conjugation by which allows $O(n)$ to act on (the left of) $\mathrm{End}(G_{m,n} \times \mathbb{R}^n)$ and $\mathrm{Hom}\,(K_{m,n}, K^{\perp}_{m,n})$. Define a vector bundle morphism λ:

$$\begin{tikzcd} O(n) \times M(n) \arrow[r, "\lambda"] \arrow[d] & \mathrm{End}\,(G_{m,n} \times \mathbb{R}^n) \cong G_{m,n} \times M(n) \arrow[d] \\ O(n) \arrow[r, "\pi"'] & G_{m,n} \end{tikzcd}$$

$\lambda(P,X) = (\pi(P), XP^{-1})$, for any $P \in O(n)$, $X \in M(n)$.

<u>Lemma 1</u>.

If λ acts on the left of $O(n) \times M(n)$ by left multiplication in each factor, then λ is equivariant.

<u>Proof</u> Let $Q \in O(n)$. Then

$$\lambda(Q.(P,X)) = \lambda(QP,QX) = (\pi(QP),\ QXP^{-1}Q^{-1})$$

$$Q.\lambda(P,X) = Q.(\pi(P),XP^{-1}) = (Q.\pi(P),QXP^{-1}Q^{-1})$$

$$= (\pi(QP),QXP^{-1}Q^{-1}), \text{ by the equivariance of } \pi.$$

$\square$

<u>Lemma 2</u>.

(i) If $P^{-1}X \in \mathfrak{h}$, then $\lambda(P,X)\circ p = p\circ\lambda(P,X)$, and $\lambda(P,X)\circ p^{\perp} = p^{\perp} \circ \lambda(P,X)$

(ii) If $P^{-1}X \in \mathfrak{m}$, then $\lambda(P,X) \circ p = p^{\perp} \circ \lambda(P,X)$, and $\lambda(P,X) \circ p^{\perp} = p \circ \lambda(P,X)$.

<u>Proof</u>. By Lemma 1, it suffices to check these when $P = \mathbb{1}_n$, noting that $\mathfrak{h}$ and $\mathfrak{m}$ are characterised as the following subspaces of $M(n)$ ([K-N] vol. 2, p. 272):

$$\mathfrak{h} = \{\begin{pmatrix} U & \mathbb{O} \\ \mathbb{O} & V \end{pmatrix} : U \in \underline{o(m)} \text{ and } V \in \underline{o(n-m)}\}$$

$$\mathfrak{m} = \{\begin{pmatrix} \mathbb{O} & W \\ -W^t & \mathbb{O} \end{pmatrix} ; W \in M_{m\times(n-m)}(\mathbb{R})\} . \qquad \square$$

Let
$$\begin{tikzcd} TO(n) \arrow[r, hook, "i"] \arrow[dr] & O(n) \times M(n) \arrow[d] \\ & O(n) \end{tikzcd}$$
be the natural (equivariant) embedding, and define the vector bundle morphism $\Pi = p^{\perp} \circ \lambda \circ i$:

$$\begin{tikzcd} TO(n) \arrow[r, "\Pi"] \arrow[d] & \mathrm{Hom}(K_{m,n},K_{m,n}^{\perp}) \arrow[d] \\ O(n) \arrow[r, "\pi"'] & G_{m,n} \end{tikzcd}$$

Finally, by restricting $d\pi$ to its horizontal distribution ($(\ker d\pi)^{\perp}$), factor Π through $d\pi$ to obtain the isomorphism B:

$$\begin{tikzcd}
 & TG_{m,n} \arrow[dr, "B"] \arrow[ddr, dashed] & \\
TO(n) \arrow[ur, "d\pi"] \arrow[rr, "\Pi"'] \arrow[d] & & \mathrm{Hom}(K_{m,n}, K^{\perp}_{m,n}) \arrow[d] \\
O(n) \arrow[rr, "\pi"'] & & G_{m,n}
\end{tikzcd}$$

<u>Remark 4.</u> B is equivariant, since everything else is !

<u>Proposition 2.</u>

The two definitions of B agree.

<u>Proof</u>. We use "B" to denote the "informally" defined isomorphism. By implicit assumption B is equivariant, whence so also is $B \circ d\pi : TO(n) \to \mathrm{Hom}(K_{m,n}, K^{\perp}_{m,n})$, which is thereby determined by its restriction to $\mathfrak{o}(n)$. So, letting $X \in \mathfrak{o}(n)$, we have that $B \circ d\pi(X) = B(\frac{d}{dt}|_o \, \pi(\exp tX)) = B(\frac{d}{dt}|_o \, V(t))$, where $V(t) = \pi(\exp tX) = \exp tX.V_{can}$ is a curve in $G_{m,n}$ through V_{can}. ?
Now, the orthogonal rotation of V_{can} onto $V(t)$ is represented by the orthogonal matrix $\exp tX_{\mathfrak{m}}$ (where $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$ is the decomposited into $\mathfrak{h}$- and $\mathfrak{m}$-components), so that:

$$\mathcal{L}(V_t) : (K_{m,n})_{V_{can}} \to (K^{\perp}_{m,n})_{V_{can}}; \; (V_{can}, v) \to (V_{can}, v(t)^{\perp})$$

$$= (V_{can}, \sum_{j=m+1}^{n} \langle \exp tX_{\mathfrak{m}}.v, e_j\rangle e_j)$$

$$B \circ d\pi(X)(V_{can}, v) = (V_{can}, \frac{d}{dt}|_o v(t)^{\perp}) = (V_{can}, \frac{d}{dt}|_o \sum_{j=m+1}^{n} \langle \exp tX_{\mathfrak{m}}.v, e_j\rangle e_j)$$

$$= (V_{can}, \sum_{j=m+1}^{m} \langle X_{\mathfrak{m}}.v, e_j\rangle e_j) = (V_{can}, X_{\mathfrak{m}}.v)$$

$$= p^{\perp} \circ \lambda(\mathbb{1}, X_{\mathfrak{m}})(V_{can}, v) = p^{\perp} \circ \lambda \circ i(X_{\mathfrak{m}})(V_{can}, v) = \Pi(X_{\mathfrak{m}})(V_{can}, v).$$

$\square$

Proposition 3.

B is a connection-preserving isometry.

Proof. Since π is a Riemannian submersion, to show that B is an isometry it suffices that $\Pi|(\ker d\pi)^{\perp}$ be an isometry. Now, $\Pi = p^{\perp} \circ \lambda \circ i$, with i an isometric inclusion,

$$\begin{array}{ccccc} O(n) \times M(n) & \xrightarrow{\lambda} & G_{m,n} \times M(n); & & (P,X) \mapsto (\pi(P), XP^{-1}) \\ \downarrow & & \downarrow & & \\ O(n) & \xrightarrow{\pi} & G_{m,n} & & \end{array}$$

clearly an isometry on fibres, and $p^{\perp}$ isometric on those endomorphisms mapping into $K_{m,n}^{\perp}$.

To show that B preserves connections, we note that

$$\nabla\Pi(\mathfrak{m},\mathfrak{m}) = \nabla B(d\pi(\mathfrak{m}), d\pi(\mathfrak{m})) + B\nabla d\pi(\mathfrak{m},\mathfrak{m}) = \nabla B(d\pi(\mathfrak{m}), d\pi(\mathfrak{m}))$$

by Corollary 2.5.1 (ii).

Thus, by equivariance of Π and B:

B is connection-preserving ($\nabla B = 0$) iff $\nabla\Pi(\mathfrak{m},\mathfrak{m}) = 0$.

Let $X, Y \in \mathfrak{m}$, and let Y_L be the left-invariant vector field extending Y. Then:

$$\nabla_X \Pi(Y) = \nabla_X(\Pi \circ Y_L) - \Pi(\nabla_X Y_L).$$

(1) $\underline{\Pi(\nabla_X Y_L) = 0}$

The Levi-Civita connection of a left-invariant, Ad-invariant metric on a Lie group G is characterised on left-invariant vector fields by ([K-N] vol. 2, p. 197):

$$\nabla_{X_L} Y_L = \frac{1}{2}[X_L, Y_L] = \frac{1}{2}[X,Y]_L, \text{ for } X,Y \in \mathfrak{g}$$

Since in addition $G_{m,n}$ is a symmetric space, we have $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, and hence:

$$\Pi(\nabla_X Y_L) = B \circ d\pi(\nabla_X Y_L) = \frac{1}{2} B \circ d\pi([X,Y]) = 0.$$

(2) $\underline{\nabla_X(\Pi \circ Y_L) = 0}$

We think of $\Pi \circ Y_L$ as a section of $\pi^{-1}\operatorname{Hom}(K_{m,n}, K^{\perp}_{m.n}) \cong$

$$\operatorname{Hom}(\pi^{-1}K_{m,n}, \pi^{-1} K^{\perp}_{m,n}).$$

If $v: O(n) \to \mathbb{R}^n$ is such that $\lambda(\pi(P), v(P))$ is a local section of $\pi^{-1}K_{m,n}$ about $\mathbb{1}_n$, we have:

$$\nabla_X(\Pi \circ Y_L)(V_{can}, v(\mathbb{1})) = \nabla_X(\Pi \circ Y_L \circ (\pi, v)) - \Pi(Y)(\nabla_X(\pi, v)).$$

(a) By Remark 2, $\nabla_X(\pi, v) = p(V_{can}, dv(\mathbb{1})(X))$. Thus:

$$\Pi(Y)(\nabla_X(\pi,v)) = p^{\perp} \circ \lambda(\mathbb{1}, Y) \circ p(V_{can}, dv(\mathbb{1})(X))$$

$$= p^{\perp} \circ \lambda(\mathbb{1}, Y)(V_{can}, dv(\mathbb{1})(X)), \text{ by Lemma 2 (ii).}$$

(b) $\Pi \circ Y_L \circ (\pi, v)$ is a local section of $\pi^{-1} K^{\perp}_{m,n}$:

$$\Pi \circ Y_L \circ (\pi,v)(P) = p^{\perp} \circ \lambda(P, PY)(\pi(P), v(P)) = p^{\perp}(\pi(P), PYP^{-1}.v(P))$$

$$= p^{\perp}(\pi(P), PYP^{t}.v(P)).$$

$$\nabla_X(\Pi \circ Y_L \circ (\pi,v)) = p^{\perp}(V_{can}, d(PYP^t.v(P))(\mathbb{1})(X)), \text{ by Remark 2.}$$

$$= p^{\perp}(V_{can}, XY.v + YX^t.v + Ydv(\mathbb{1})(X))$$

$$= p^{\perp}(V_{can}, [X,Y].v + Ydv(\mathbb{1})(X)), \text{ since } X^t = -X.$$

$$= p^{\perp} \circ \lambda(\mathbb{1},[X,Y])(V_{can}, v(\mathbb{1})) + p^{\perp} \circ \lambda(\mathbb{1},Y)(V_{can}, dv(\mathbb{1})(X))$$

$$= p^{\perp} \circ \lambda(\mathbb{1},Y)(V_{can}, dv(\mathbb{1})(X)), \text{ by Lemma 2(i), since}$$

$$[X,Y] \in \mathfrak{h} \text{ and } (\pi,v) = p \circ (\pi,v). \qquad \square$$

<u>Proposition 4.</u>

$$C(\nabla_X d\phi) = d\gamma_\phi(X), \text{ for all } X \in TM.$$

<u>Proof</u>. We show that $A(\nabla_X d\phi) = B(d\gamma_\phi(X))$:

$$\begin{tikzcd}
\mathrm{Hom}(TM,TM^{\perp}) \arrow[r, "A"] \arrow[d] & \mathrm{Hom}(K_{m,n},K^{\perp}_{m,n}) \arrow[d] & TG_{m,n} \arrow[l, "B"'] \arrow[dl] & TO(n) \arrow[l, "d\pi"'] \arrow[d] \arrow[ll, bend right, "\Pi"'] \\
M \arrow[r, "\gamma_\phi"'] & G_{m,n} & & O(n) \arrow[ll, "\pi"]
\end{tikzcd}$$

Let $x_o \in M$, and $X = x'(0) \in T_{x_o}M$; we may assume without loss of generality that $\phi(x_o) = 0$ and $\gamma_\phi(x_o) = V_{can}$. Now, π is a fibre bundle (over paracompact base) and thereby admits a path-lifting function ([Spa] p. 96). Thus, the path $\gamma_\phi \circ x(t)$ is $G_{m,n}$ may be lifted to a path $P(t)$ in $O(n)$, with $P(0) = \mathbb{1}$:

$$\begin{array}{ccc} (M,x_o) & \xrightarrow{\gamma_\phi} & (G_{m,n}, V_{can}) \\ x(t)\uparrow & & \uparrow \pi \\ (\mathbb{R},0) & \xrightarrow[P(t)]{} & (O(n), \mathbb{1}) \end{array}$$

If $(y(t), Q(t))$ is another such pair (with $y'(0) = X$):

$$d\pi(P'(0)) = d\gamma_\phi(x'(0)) = d\gamma_\phi(X) = d\gamma_\phi(y'(0)) = d\pi(Q'(0)) \ldots(\dagger)$$

Let $v_o \in V_{can}$, and extend to a map $v:\mathbb{R}^n \to \mathbb{R}^n$ satisfying:

$$v(\phi \circ x(t)) = P(t).v_o \qquad \ldots(\ddagger)$$

(cf [K-N] vol. 1, Appendix 3). Since ϕ is locally 1-1, the vector field Γ_v (= graph v) on $\mathbb{R}^n$ may be locally pulled-back to M; denote by Y the local pullback of Γ_v to a neighbourhood of $x_o \in M$:

$$\begin{tikzcd} & TM \arrow[r, "d\phi"] \arrow[d] & T\mathbb{R}^n \arrow[d, shift left] \\ (U,x_o) \subset M \arrow[ur, "Y"] \arrow[rr] & & \mathbb{R}^n \arrow[u, bend right, "\Gamma_v"'] \end{tikzcd}$$

We note that, $I_1(Y(x_o)) = (V_{can}, v_o) \qquad \ldots (\perp)$

(i) $B(d\gamma_\phi(X)) \underset{(\dagger)}{=} B(d\pi(P'(0))) = \mathrm{II}(P'(0))$. Thus:

$$[B(d\gamma_\phi(X))]\ (V_{can},v_o) = p^\perp \circ \lambda(\mathbb{1},P'(0))(V_{can},v_o) = p^\perp(V_{can}, P'(0).v_o)$$

(ii) $[A(\nabla_X d\phi)](V_{can}, v_o) = I_2 \circ \nabla_X d\phi \;\; o I_1^{-1}(V_{can}, v_o) \underset{(\ddagger)}{=} I_2(\nabla_X d\phi(Y))$

$$\underset{\text{Cor. 2.5.1(i)}}{=} \quad p^{\perp} \circ I(\nabla_X(\phi^{-1}\Gamma_v)) = p^{\perp} \circ I(\phi^{-1}({}^{\mathbb{R}^n}\nabla_{d\phi(X)}\Gamma_v))$$

$$= p^{\perp}(V_{can}, dv(\phi x_o)(d\phi(X))) = p^{\perp}(V_{can}, \frac{d}{dt}\Big|_o \, v \circ \phi \circ x(t))$$

$$\underset{(\ddagger)}{=} p^{\perp}(V_{can}, \frac{d}{dt}\Big|_o \, P(t).v_o) = p^{\perp}(V_{can}, P'(0).v_o). \qquad \square$$

§2. THE GAUSS SECTION OF A RIEMANNIAN IMMERSION

Suppose now that (N,h) is an arbitrary n-dimensional Riemannian manifold, and $\phi:(M,g) \to (N,h)$ a *Riemannian immersion*. To generalize Theorem 1.1, let $\sigma: G_m N \to N$ be the m-plane *Grassmann bundle* of N, where fibre over $y \in N$ is the collection of all m-planes in $T_y N$, and define γ_ϕ to be the *Gauss section* of ϕ:

$$\begin{tikzcd} & G_m N \arrow[d, "\sigma"] \\ M \arrow[ur, "\gamma_\phi"] \arrow[r, "\phi"'] & N \end{tikzcd} \qquad ; \quad \gamma_\phi(x) = d\phi(T_x M)$$

<u>Remark 1</u>. γ_ϕ is a section of $\phi^{-1}(G_m N)$. ady over M

<u>Remark 2</u>. When $N = \mathbb{R}^n$, the Gauss section is the graph of the Gauss map.

To geometrize $G_m N$ we note the following bicorrespondences:

An m-plane W in $T_y N$ $\longleftrightarrow$

The equivalence class of all orthonormal frames in $T_y N$ whose first m vectors span W $\longleftrightarrow$ The orbit of any such frame under the subgroup $O(m) \times O(n-m)$ of $O(n)$.

Thus, if $\xi : O(N) \to N$ is the principal $O(n)$-bundle of orthonormal frames of (N,h), G_mN is isomorphic to the orbit space $O(N)/O(m) \times O(n-m)$, which is in turn isomorphic to the associated homogeneous fibre bundle $O(N) \times_{O(n)} G_{m,n}$ ([Hus] p.70). Now G_mN may be given a metric through the generalities of Chapter 2 §1, using the Levi-Civita connection in $O(N)$ and the canonical $O(n)$-invariant metric on $G_{m,n}$.

<u>Remark 3</u>. $O(N)$ fibres over G_mN as a principal $O(m) \times O(n-m)$-bundle:

$$\begin{tikzcd} O(N) \arrow[d, "\xi"'] \arrow[dr, "\chi"] & \\ N & G_mN \arrow[l, "\sigma"] \end{tikzcd}$$

The metric on G_mN is then that unique metric w.r.t. which χ is a Riemannian submersion.

To generalize §1, we consider the *fibre product:*

$$\begin{tikzcd} & \mathfrak{T} = G_mN \oplus TN = \mathfrak{G} \arrow[dl, "\beta"'] \arrow[d] \arrow[dr, "\alpha"] & \\ TN \arrow[r, "\tau_N"'] & N & G_mN \arrow[l, "\sigma"] \end{tikzcd}$$

Since $\mathfrak{G} | (G_mN)_y = (G_mN)_y \times T_yN$ (Chapter 2 §3), $\mathfrak{G}$ is a vector bundle of rank n, with the *canonical* and *ortho-complement* bundles of G_mN as vector sub-bundles:

$$K_mN = \{(W,w) \in \mathfrak{G} : w \in W\},\ K_m^{\perp}N = \{(W,w) \in \mathfrak{G} : w \perp W\}$$

Then $\mathfrak{G} = K_mN \oplus K_m^{\perp}N$ so that, as in §1, K_mN and $K_m^{\perp}N$ inherit the geometry of $\mathfrak{G}$ (see Chapter 2 §3) via the projection morphisms $p:\mathfrak{G} \to K_mN$ and $p^{\perp}:\mathfrak{G} \to K_m^{\perp}N$; for example, $(TK_mN)^H_{(W,w)} = dp(T\mathfrak{G})^H_{(W,w)}$.

<u>Proposition 1</u>

$\mathfrak{G}$ (and hence K_mN, $K_m^{\perp}N$) is a Riemannian vector bundle.

<u>Proof</u>. The fibre metric in $\mathfrak{G}$ is defined by requiring β to be an isometry on fibres: $\|(W,w)\|^2 = k(w,w)$. Let $W(t)$ be any curve in G_mN. Then, by Proposition 2.3.1 (i), parallel translation along $W(t)$ is an isometry:

$$\|\alpha_t^W(W(0),w)\|^2 = \|(W(t),\ (\tau_N)_t^{\sigma\circ W}(w))\|^2 = h((\tau_N)_t^{\sigma\circ W}(w),(\tau_N)_t^{\sigma\circ W}(w))$$

$$= h(w,w) = \|(W(0),w)\|^2. \qquad \square$$

Define the vector bundle fibre isomorphism $\mathcal{I}$:

$$\begin{array}{ccc} \phi^{-1}(TN) & \xrightarrow{\ \mathcal{I}\ } & \mathfrak{G} \\ \downarrow & & \downarrow \\ M & \xrightarrow[\ \gamma_\phi\ ]{} & G_mN \end{array} \quad ;\ \mathcal{I}(x,w) = (\gamma_\phi(x),w)$$

and restrict to TM (resp. $TM^{\perp}$) to obtain the morphism $\mathcal{I}_1$ (resp. $\mathcal{I}_2$):

$$\begin{tikzcd} TM \arrow[r, "\mathcal{I}_1"] \arrow[d] & K_mN \arrow[d] \\ M \arrow[r, "\gamma_\phi"'] & G_mN \end{tikzcd} \qquad \begin{tikzcd} TM^{\perp} \arrow[r, "\mathcal{I}_2"] \arrow[d] & K_m^{\perp}N \arrow[d] \\ M \arrow[r, "\gamma_\phi"'] & G_mN \end{tikzcd}$$

Conjugating by $\mathcal{I}$ gives a morphism $\mathbb{A}$:

$$\begin{tikzcd} \mathrm{End}\ \phi^{-1}(TN) \arrow[r, "\mathbb{A}"] \arrow[d] & \mathrm{End}\ \mathbb{G} \arrow[d] \\ M \arrow[r, "\gamma_\phi"'] & G_mN \end{tikzcd}$$

which when restricted to Hom $(TM, TM^{\perp})$ maps into Hom $(K_mN, K_m^{\perp}N)$; $\mathbb{A}\mathcal{L} = \mathcal{I}_2 \circ \mathcal{L} \circ \mathcal{I}_1^{-1}$, for any $\mathcal{L} \in$ Hom $(TM, TM^{\perp})$. We note that End $\mathbb{G} \cong G_mN \oplus TN$, when the latter is viewed as a fibration over G_mN. In this case we shall employ the following notation:

$$\begin{tikzcd}
 & & O(N) \times M(n) \arrow[ddll] \arrow[d, "\eta'"] & & \\
 & & \mathrm{End}\ TN \arrow[ddll, dashed, "\tau'"'] & & \\
O(N) \arrow[d, "\xi"'] & & & & O(N) \times G_{m,n} \times M(n) \arrow[uull] \arrow[llll] \arrow[d, "\nu'"] \arrow[ddll] \\
N & & & & G_mN \oplus \mathrm{End}\ TN \arrow[uull, dashed, "\beta'"'] \arrow[llll, dashed, "\rho'"'] \arrow[ddll, "\alpha"] \\
 & & O(N) \times G_{m,n} \arrow[uull] \arrow[d, "\mu"] & & \\
 & & G_mN \arrow[uull, "\sigma"] & &
\end{tikzcd}$$

with the corresponding unprimed symbols being reserved for use in the analogous diagram for $G_mN \oplus TN$.

Proposition 2

$$\begin{array}{ccc} \mathrm{Hom}\,(TM, TM^{\perp}) & \xrightarrow{\;\;A\;\;} & \mathrm{Hom}\,(K_mN,\, K_m^{\perp}N) \\ \downarrow & & \downarrow \\ M & \xrightarrow[\gamma_\phi]{} & G_mN \end{array}$$

is a connection-preserving fibre isometry.

Proof. With the fibre metric in End $\mathfrak{G}$ such that β' maps fibres of α isometrically onto fibres of τ', it is clear that A is a fibre isometry.

We show that I preserves parallel translation, from which it follows that so also do I_1 and I_2, and hence A (Proposition 2.4.1).

Let $x(t)$ be a path in M, and $w_0 \in T_{\phi(x_0)}N$, so that $(x_0, w_0) \in \phi^{-1}(TN)_{x_0}$. Parallel translation in $\phi^{-1}(TN)$ is characterised:

$$(\tilde{\tau}_N)_t^x\,(x_0, w_0) = (x(t),\ (\tau_N)_t^{\phi\circ x}(w_0)), \quad \text{so that}$$

$$I(\tilde{\tau}_N)_t^x(x_0, w_0) = (\gamma_\phi \circ x(t),\ (\tau_N)_t^{\phi\circ x}(w_0)).$$

On the other hand, by Proposition 2.3.1 (i):

$$\alpha'{}_t^{\gamma_\phi\circ x}\,(I(x_0, w_0)) = \alpha'{}_t^{\gamma_\phi\circ x}\,(\gamma_\phi(x_0), w_0) = (\gamma_\phi\circ x(t), (\tau_N)_t^{\phi\circ x}(w_0)).$$

□

In order to generalize the definition of B in §1, we consider the fibre product:

$$\begin{tikzcd} O(N) \arrow[d, "\xi"'] & O(N) \oplus \mathrm{End}\, TN \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] \\ N & \mathrm{End}\, TN \arrow[l, "\tau'"] \end{tikzcd}$$

where π_i is the projection onto the i^{th} factor. Now, the fibre model of $O(N) \oplus TN$ is $O(n) \times M(n)$, and that of $G_m N \oplus \mathrm{End}\, TN$ is $G_{m,n} \times M(n)$, so that the equivariant map $\lambda: O(n) \times M(n) \to G_{m,n} \times M(n)$, together with the identity morphism of $O(N) \to N$, provides an associated morphism Λ (cf [Eel] p. 57):

$$\begin{tikzcd} O(N) \oplus \mathrm{End}\, TN \arrow[rr, "\Lambda"] \arrow[dr, "\xi \circ \pi_1"'] & & G_m N \oplus \mathrm{End}\, TN \arrow[dl, "\rho'"] \\ & N & \end{tikzcd}$$

Recalling the fibering χ of Remark, Λ is in fact a χ-morphism:

$$\begin{tikzcd} O(N) \oplus \mathrm{End}\, TN \arrow[r, "\Lambda"] \arrow[d, "\pi_1"'] & G_m N \oplus \mathrm{End}\, TN \arrow[d, "\alpha'"] \\ O(N) \arrow[r, "\chi"] \arrow[dr, "\xi"'] & G_m N \arrow[d, "\sigma"] \\ & N \end{tikzcd}$$

By regarding $TO(N)^V$ (= ker $d\xi$) as a bundle over N, with fibre $TO(n)$:

$$\begin{tikzcd} TO(N)^V \arrow[r, "\tau_{O(N)}"] \arrow[dr] & O(N) \arrow[d, "\xi"] \\ & N \end{tikzcd}$$

we define I to be the morphism associated to $i: TO(n) \to O(n) \times M(n)$:

$$\begin{tikzcd} TO(N)^V \arrow[rr, "I"] \arrow[dr] & & O(N) \oplus \mathrm{End}\, TN \arrow[dl] \\ & N & \end{tikzcd}$$

We then have the morphism $\mathfrak{P} = p^{\perp} \circ \Lambda \circ I$:

$$\begin{tikzcd} TO(N)^V \arrow[r, "\mathfrak{P}"] \arrow[d] & \mathrm{Hom}\,(K_mN, K_m^{\perp}N) \arrow[d] \\ O(N) \arrow[r, "\chi"'] & G_mN \end{tikzcd}$$

<u>Lemma 1</u>.

Restricting to the fibre over $y \in N$ in the appropriate bundle:

(i) $\Lambda = \nu_E' \circ \lambda \circ (\xi_E^{-1} \times (\eta_E')^{-1}) = (\mu_E \times \eta_E') \circ \lambda \circ (\xi_E^{-1} \times (\eta_E')^{-1})$.

(ii) $I = (\xi_E \times \eta_E') \circ d\xi_E^{-1}$.

(iii) $\mathfrak{P} = \nu_E' \circ \Pi \circ d\xi_E^{-1}$

for any $E \in \xi^{-1}(y)$. □

Finally, writing $\mathfrak{V}$ = ker $d\chi$ and $\mathfrak{H} = (\ker\,(d\chi)^V)^{\perp}$, we have (cf. Fig. 4):

$TO(N)^V = \ker d\xi = \mathfrak{V} \oplus \mathcal{H}$ and $TO(N)^H = (\ker d\xi)^\perp = (\ker (d\chi)^H)^\perp$

so that, by restriction to $\mathcal{H}$, $\mathcal{P}$ may be factored through $d\chi$ to give the isomorphism $\mathcal{B}$:

$$\begin{tikzcd}
& (TG_mN)^V \arrow[dr, "\mathcal{B}"] \arrow[ddr, dashed] & \\
TO(N)^V \arrow[ur, "d\chi"] \arrow[rr, "\mathcal{P}"'] \arrow[d] & & \mathrm{Hom}\ (K_mN,\ K_m^\perp N) \arrow[d] \\
O(N) \arrow[rr, "\chi"'] & & G_mN
\end{tikzcd}$$

Proposition 3.

$\mathcal{B}$ is a connection-preserving isometry.

Proof. Restricting to the fibre of $(TG_mN)^V$ over $W \in G_mN$, we have from Lemma 1 that $\mathcal{B} = \nu_E' \circ B \circ d\mu_E^{-1}$, for any $E \in \chi^{-1}(W)$, whence, by Proposition 1.3, $\mathcal{B}$ is an isometry.

Since $O(N)$ fibres over both N and G_mN, the connection in $\mathcal{H} \to O(N)$ (viz. the $\mathcal{H}$-projection of the Levi-Civita connection of (N,h)) splits into three complementary partial connections:

$$\begin{aligned}
(T\mathcal{H})^H &= (T\mathcal{H})^{HH} \oplus (T\mathcal{H})^{HV} \\
&= (T\mathcal{H})^{HH} \oplus (T\mathcal{H})^{H\mathcal{H}} \oplus (T\mathcal{H})^{H\mathfrak{V}}
\end{aligned}$$

To help keep track of these in the sequel, we have the following diagram:

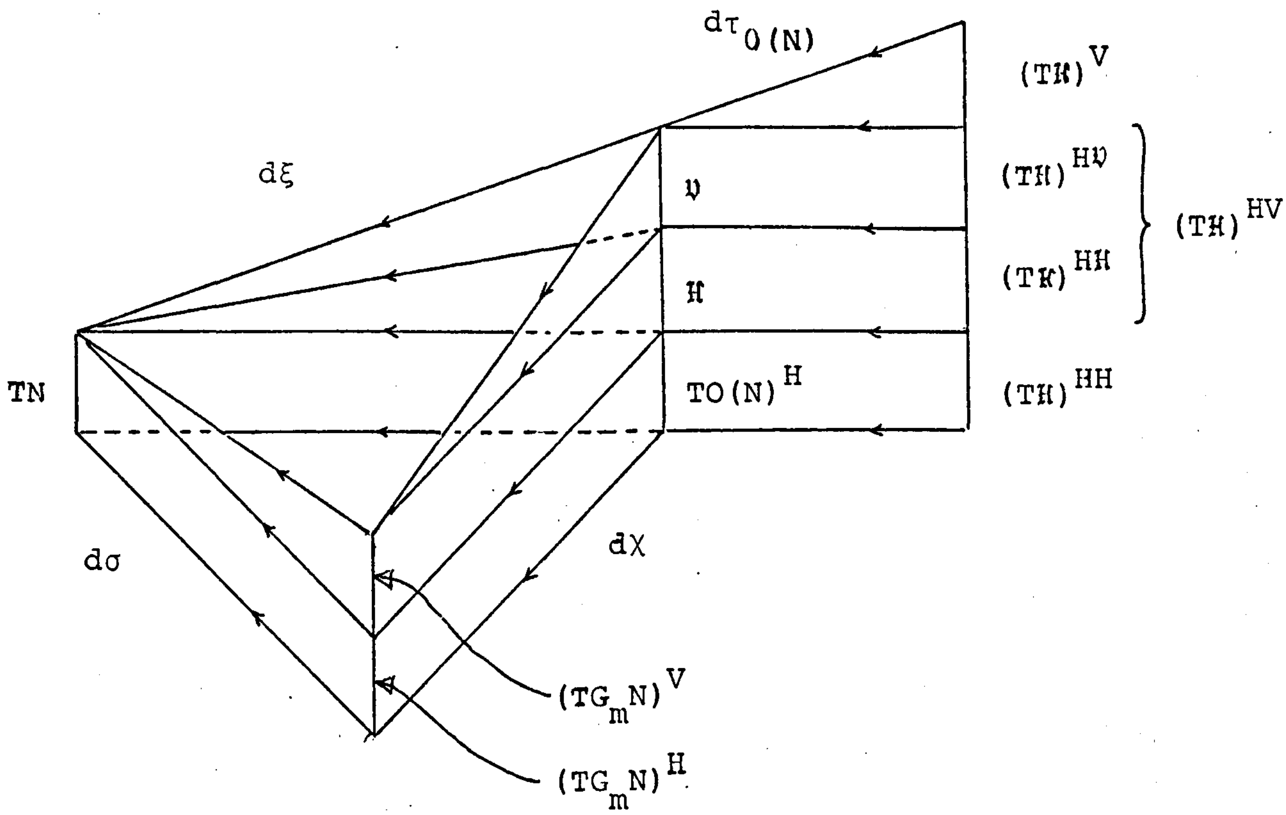

Figure 4

Since $(TG_mN)^V = d\chi(\mathcal{H})$, to show that $\mathcal{B}$ preserves connections it is sufficient that $\nabla\mathcal{B}(d\chi(\mathcal{H})) = 0$. Because $TG_mN = d\chi(\mathcal{H}) \oplus d\chi(TO(N)^H)$ (i.e. we may ignore the χ-vertical distribution $\mathfrak{V}$), there are two cases to verify:

(a) $\nabla\mathcal{B}(d\chi(X), d\chi(Y)) = 0$, for $X,Y \in \mathcal{H}$.

(b) $\nabla\mathcal{B}(d\chi(X), d\chi(Y)) = 0$, for $X \in TO(N)^H$ and $Y \in \mathcal{H}$.

Now, $\nabla\mathcal{B}(d\chi(X), d\chi(Y)) = \nabla\mathcal{P}(X,Y) - \mathcal{B}\nabla d\chi(X,Y)$.

(1)(a) $\nabla d\chi(\mathcal{H},\mathcal{H}) = 0$ by Corollary 2.5.1 (ii), since χ is a Riemannian submersion (Remark 3) and $\mathcal{H}$ lies in the χ-horizontal distribution of O(N).

(b) Similarly, $\nabla d\chi((TO(N)^H,\mathcal{H}) = 0$.

(2)(a) To show $\nabla\mathcal{P}(\mathcal{H},\mathcal{H}) = 0$ we note that if X, Y are ξ-vertical, then by Lemma 1 (iii):

$$\nabla_X\mathcal{P}(Y) = (\nabla_X\nu_E') \circ \Pi \circ d\xi_E^{-1}(Y) + \nu_E' \circ \nabla_X\Pi \circ d\xi_E^{-1}(Y)$$
$$+ \nu_E' \circ \Pi \circ \nabla_X\, d\xi_E^{-1}(Y)$$

(i) ξ_E is totally geodesic (being an isometry of O(n) with the fibre of O(N) through E), so that $\nabla d\xi_E = 0$.

(ii) $\Pi = B \circ d\pi$, where π is a Riemannian submersion and B is connection-preserving (Proposition 1.3), so that $\nabla\Pi((\ker d\pi)^\perp, (\ker d\pi)^\perp) = 0$ by Corollary 2.5.1 (ii).

We note that $\mathcal{H}_E = d\xi_E\,(\ker d\pi)^\perp$.

(iii) $\nu_E' = \mu_E \times \eta_E'$, so that $d\nu_E' : \overline{TG}_{m,n} \to \overline{T\sigma^{-1}(y)} = (T\,\mathrm{End}\,\mathcal{G})^{HV}\restriction\sigma^{-1}(y)$

Thus, $d\nu_E'$ is horizontal and hence $\nabla\nu_E' = 0$.

(b) To show $\nabla\mathcal{P}(TO(N)^H,\mathcal{H}) = 0$ is equivalent to showing

$d\mathcal{P}(T\mathcal{H})^{HH} \subset (T\,\mathrm{Hom}(K_mN,K_m^\perp N))^{HH}$ (indeed, $\nabla\mathcal{P}(\mathcal{H},\mathcal{H}) = 0$ is equivalent to $d\mathcal{P}(T\mathcal{H})^{HH} \subset (T\,\mathrm{Hom}(K_mN,K_m^\perp N))^{HV}$), which in turn is equivalent to showing that $\mathcal{P}$ preserves parallel translation of $\mathcal{H}$ along ξ-horizontal paths in O(N).

Let E(t) be a ξ-horizontal path in O(N), with $\xi\circ E(t) = y(t)$; then $W(t) = \chi\circ E(t)$ is a horizontal path in G_mN (Remark 3, Corollary 2.5.1 (ii)). Let F(t) be a path in $\xi^{-1}(y_0)$ with $F(0) = E(0)$ and $F'(0) = Y \in \mathcal{H}$. Then, by Proposition 2.1.1 (i):

$$(\tau_{O(N)})_t^E(Y) = \frac{d}{ds}\Big|_0\, \xi_t^y\,(F(s)).$$

We note that $(\tau_{0(N)})_t^E$ preserves $\mathcal{H}$ (in this case), and so coincides with the parallel translation of $\mathcal{H}$. Now:

$$\mathcal{P}(\tau_{0(N)})_t^E(Y) = p^{\perp} \circ \nu'_{E(t)} \circ \Pi \circ d\xi_{E(t)}^{-1}\left(\frac{d}{ds}\Big|_0 \xi_t^Y \circ F(s)\right),$$

by Lemma 1 (iii)

$$= p^{\perp} \circ \nu'_{E(t)} \circ \Pi\left(\frac{d}{ds}\Big|_0 \xi_{E(t)}^{-1} \circ \xi_t^Y \circ F(s)\right)$$

$$= p^{\perp} \circ \nu'_{E(t)} \circ \Pi\left(\frac{d}{ds}\Big|_0 \xi_E^{-1} \circ F(s)\right),$$ by the ξ-horizontality of $E(t)$

$$= p^{\perp} \circ \nu'_{E(t)} \circ \Pi(d\xi_E^{-1}(Y)).$$

On the other hand, since $W(t)$ is horizontal we have that (Corollary 2.3.1):

$$\alpha'^W_t = \rho'^Y_t = \nu'_{E(t)} \circ (\nu'_E)^{-1}$$ so that:

$$\alpha'^W_t(\mathcal{P}(Y)) = \nu'_{E(t)} \circ (\nu'_E)^{-1} \circ \nu'_E \circ \Pi(d\xi_E^{-1}(Y)) = \nu'_{E(t)} \circ \Pi(d\xi_E^{-1}(Y)).$$

The equality $\mathcal{P}(\tau_{0(N)})_t^E(Y) = \alpha'^W_t(\mathcal{P}(Y))$ now follows by noting that parallel translation in Hom $(K_mN, K_m^{\perp}N)$ is obtained by projecting α'^W_t onto $K_m^{\perp}N$ (cf. Proposition 2.4.1). □

<u>Corollary 1.</u>

$$\begin{array}{ccc} \mathrm{Hom}\,(TM, TM^{\perp}) & \xrightarrow{\mathcal{C} = \mathcal{B}^{-1} \circ \mathcal{A}} & (TG_mN)^V \\ \downarrow & & \downarrow \\ M & \xrightarrow[\gamma_\phi]{} & G_mN \end{array}$$

is a connection-preserving fibre isometry. □

The following concerns the stability under parallel translation of the relations (between TN and G_mN) of "$\in$" and "$\perp$":

Lemma 2

If $y(t)$ is a path in N and $w_0 \in T_{y_0}N$, $W_0 \in (G_mN)_{y_0}$, then:

(i) $w_0 \in W_0$ iff $(\tau_N)_t^y(w_0) \in \sigma_t^y(W_0)$

(ii) $w_0 \perp W_0$ iff $(\tau_N)_t^y(w_0) \perp \sigma_t^y(W_0)$.

Proof. Formally, the relations "$\in$" and "$\perp$" in TN $\times$ G_mN are defined in terms of the corresponding relations between the fibre models. Thus:

$$\left.\begin{array}{l} w_0 \in W_0 \text{ iff } \eta_E^{-1}(w_0) \in \mu_E^{-1}(W_0) \\ w_0 \perp W_0 \text{ iff } \eta_E^{-1}(w_0) \perp \mu_E^{-1}(W_0) \end{array}\right\} \begin{array}{l} \text{for any} \\ E \in \xi^{-1}(y_0). \end{array}$$

Thus:

$$(\tau_N)_t^y(w_0) \in \sigma_t^y(W_0) \text{ iff } (\eta_{\xi_t^y(E)})^{-1}(\tau_N)_t^y(w_0) \in (\mu_{\xi_t^y(E)})^{-1}\sigma_t^y(W_0)$$

$$\text{iff } \eta_E^{-1}(w_0) \in \mu_E^{-1}(W_0) \text{ iff } w_0 \in W_0. \quad \square$$

Proposition 4

$$\mathfrak{C}(\nabla_X d\phi) = (d\gamma_\phi)^V(X), \text{ for all } X \in TM.$$

Proof. (cf. Proof of Proposition 1.4).

We show that $\mathfrak{A}(\nabla_X d\phi) = \mathfrak{B}((d\gamma_\phi)^V(X))$.

Let $x_0 \in M$, and $X = x'(0) \in T_{x_0}M$. Put $W_0 = \gamma_\phi(x_0)$, and choose $E \in \xi^{-1}(\phi(x_0))$ such that $W_0 = \mu_E(V_{can})$. Since G_mN is certainly Hausdorff and paracompact (assuming N paracompact),

χ is a Hurewicz fibration ([Spa] p. 96) and so admits a path-lifting function, whereby the path $\gamma_\phi \circ x(t)$ may be lifted to a path $E(t)$ with $E(0) = E$:

$$\begin{array}{ccc} G_mN & \xleftarrow{\gamma_\phi \circ x(t)} & \mathbb{R} \\ \sigma\big\downarrow & \nwarrow^{\chi} & \big\downarrow E(t) \\ N & \xleftarrow[\xi]{} & O(N) \end{array}$$

Claim 1. $E^V(t)$ is a lift of $(\gamma_\phi \circ x)^V(t)$ (see Chapter 2 §2).

Proof. Since $d\chi\colon TO(N)^H \to (TG_mN)^H$ (cf. Fig. 4), χ commutes with parallel translation so that:

$$\chi \circ E^V(t) = \chi \circ (\xi_t^{\phi\circ x})^{-1} \circ E(t) = (\sigma_t^{\phi\circ x})^{-1} \circ \chi \circ E(t) = (\chi\circ E)^V(t)$$

$$= (\gamma_\phi \circ x)^V(t). \qquad \square$$

Claim 2. $\gamma_\phi \circ x(t) = \mu_{E(t)} \circ \mu_E^{-1}(W_0)$.

Proof. Since χ is the morphism associated to $\pi\colon O(n) \to G_{m,n}$, we have that:

$$\chi|\xi^{-1}(y) = \mu_E \circ \pi \circ \xi_E^{-1}, \text{ for any } E \in \xi^{-1}(y). \text{ Thus:}$$

$$\gamma_\phi \circ x(t) = \chi\circ E(t) = \mu_{E(t)} \circ \pi \circ \xi_{E(t)}^{-1}\circ E(t) = \mu_{E(t)} \circ \pi(\mathbb{1}_n)$$

$$= \mu_{E(t)}(V_{can}) = \mu_{E(t)} \circ \mu_E^{-1}(W_0). \qquad \square$$

If $(y(t), F(t))$ is another such pair (i.e. with $y'(0) = X$), then by Claim 1:

$$d\chi((F^V)'(0)) = \frac{d}{dt}\Big|_0 (\gamma_\phi \circ y)^V(t) = (d\gamma_\phi)^V(X)$$

$$= \frac{d}{dt}\Big|_0 (\gamma_\phi \circ x)^V(t) = d\chi((E^V)'(0)) \quad \ldots\ldots\ldots \quad (\dagger)$$

Let $w_0 \in W_0$ and put $v_0 = \eta_E^{-1}(w_o) \in V_{can}$. Define a local vector field w on N by extending $(\tau_N)_t^{\phi\circ x} \circ \eta_E V_{(t)} \circ \eta_E^{-1}(w_0)$ away from the image of $\phi\circ x(t)$ (cf [K-N] vol 1, Appendix 3). Then:

$$w(\phi\circ x(t)) = (\tau_N)_t^{\phi\circ x} \circ \eta_E V_{(t)} \circ \eta_E^{-1}(w_0) \quad \ldots\ldots \quad (\ddagger)$$

The extension may be made tangential to $\phi(M)$, because:

<u>Claim 3</u>. $w(\phi\circ x(t))$ is tangent to $\phi(M)$.

<u>Proof</u>. By Lemma 2 and $(\ddagger)$:

$$w(\phi\circ x(t)) \in d\phi(T_{\phi\circ x(t)}M) \text{ iff } w(\phi\circ x(t)) \in \gamma_\phi \circ x(t)$$

$$\text{iff } \eta_E V_{(t)} \circ \eta_E^{-1}(w_0) \in (\sigma_t^{\phi\circ x})^{-1}(\gamma_\phi \circ x(t))$$

Now, from Claims 1 and 2 it follows that:

$$(\sigma_t^{\phi\circ x})^{-1}(\gamma_\phi \circ x(t)) = (\gamma_\phi \circ x)^V(t) = \chi(E^V(t)) = \mu_{EV(t)} \circ \mu_E^{-1}(W_0)$$

Since $w_0 \in W_0$ it follows (as in Lemma 2) that:

$$\eta_E V_{(t)} \circ \eta_E^{-1}(w_0) \in \mu_E V_{(t)} \circ \mu_E^{-1}(W_0). \quad \square$$

Since ϕ is locally 1-1, there exists a local pullback Y of w to a neighbourhood of x_0. We note that

$$I_1(Y(x_0)) = (W_0, w_0) \quad \ldots\ldots \quad (\perp)$$

The main calculation now proceeds as follows:

$$\mathcal{B}((d\gamma_\phi)^V(X)) \underset{(\dagger)}{=} \mathcal{B}(d\chi((E^V)'(0))) = \mathcal{P}((E^V)'(0)),$$

by Proposition 2.2.1

$$= \nu_E' \circ \Pi \circ d\xi_E^{-1}(E^V)'(0), \text{ by Lemma 1 (iii)}$$

$$= \nu_E \circ \Pi \left(\frac{d}{dt}\Big|_0 \xi_E^{-1} \circ E^V(t)\right) \circ \nu_E^{-1}, \text{ by Proposition 2.4.1.}$$

Now, $\xi_E^{-1} \circ E^V(t)$ is a path in $O(n)$, say $P(t)$. Thus:

$$[\mathcal{B}((d\gamma_\phi)^V(X))](W_0,w_0) = \nu_E \circ p^\perp \circ \lambda(\mathbb{1}_n, \frac{d}{dt}\Big|_0 P(t))(V_{can}, v_0)$$

$$= \nu_E \circ p^\perp(V_{can}, \frac{d}{dt}\Big|_0 P(t).v_0) = p^\perp(W_0, \frac{d}{dt}\Big|_0 \eta_E(P(t).v_0))$$

$$= p^\perp(W_0, \frac{d}{dt}\Big|_0 \eta(E.P(t),v_0)) = p^\perp(W_0, \frac{d}{dt}\Big|_0 \eta(E^V(t),v_0))$$

$$= p^\perp \circ \mathcal{I}(W_0, \frac{d}{dt}\Big|_0 \eta_{E^V(t)} \circ \eta_E^{-1}(w_0)) \underset{(\ddagger)}{=}$$

$$\mathcal{I}_2(x_0, \frac{d}{dt}\Big|_0 ((\tau_N)_t^{\phi\circ x})^{-1}(w\circ\phi\circ x(t)))$$

$$= \mathcal{I}_2(\phi^{-1}({}^N\nabla_{d\phi(X)}w)) = \mathcal{I}_2(\nabla_X d\phi(Y)), \text{ by Corollary 2.5.1(i),}$$

$$\underset{(\downarrow)}{=} \mathcal{I}_2 \circ \nabla_X d\phi \circ \mathcal{I}_1^{-1}(W_0,w_0) = [A(\nabla_X d\phi)](W_0,w_0). \quad \square$$

<u>Corollary 2</u>.

ϕ is totally geodesic iff γ_ϕ is horizontal. $\square$

The $\phi^{-1}(TN)$-valued 1-form $d\phi$ has *Hodge-de Rham Laplacian* $-\nabla\tau(\phi)$, and *rough Laplacian* $-\text{Trace}\ \nabla^2 d\phi$, related by the *Bochner-Weitzenböck* formula:

<u>Lemma 3</u>. ([E-L] Proposition 1.34)

$$\mathrm{Ric}_\phi \circ d\phi - d\phi \circ \mathrm{Ric}^M = \nabla\tau(\phi) - \mathrm{Trace}\ \nabla^2 d\phi.$$

where

$$\mathrm{Ric}_\phi \in \mathbb{C}(\phi^{-1}(TN)^* \otimes \phi^{-1}(TN)) \quad \text{(cf. Chapter 1 §7)}$$

$$\mathrm{Ric}_\phi\ (A) = \mathrm{Trace}\ R^N(A, d\phi)d\phi, \text{ for all } A \in \mathbb{C}(\phi^{-1}(TN)). \quad \square$$

<u>Remark 4</u>. When ϕ is a Riemannian immersion, Lemma 3 is a consequence of the *equation of Gauss* ([K-N] vol 2, p. 23):

$$R^N(d\phi(X), d\phi(Y))d\phi(Z)^\top = R^M(X,Y)Z + B(X,Y)Z$$

where

$$g(B(X,Y)Z,W) = h(\nabla d\phi(X,Z), \nabla d\phi(Y,W)) - h(\nabla d\phi(Y,Z), \nabla d\phi(X,W))$$

and the *equation of Codazzi* ([K-N] vol 2, p. 25):

$$R^N(d\phi(X), d\phi(Y))d\phi(Z)^\perp = D_X(\nabla d\phi)(Y,Z) - D_Y(\nabla d\phi)(X,Z),$$

where D is the connection in the normal bundle of ϕ. For then:

$$\begin{aligned} g(\mathrm{Ric}_\phi \circ d\phi(X)^\top - \mathrm{Ric}^M(X), Y) &= \sum_i \{h(\nabla d\phi(X,E_i), \nabla d\phi(Y,E_i)) \\ &\qquad - h(\nabla d\phi(E_i,E_i), \nabla d\phi(X,Y))\} \end{aligned}$$

$$= h(\nabla_X \tau(\phi), d\phi(Y)) - \sum_i h(\nabla_{E_i} \nabla d\phi(X,E_i), d\phi(Y)) \quad \text{([K-N] vol 2,p.14).}$$

$$= g(\nabla_X \tau(\phi)^\top - \mathrm{Trace}\ \nabla^2 d\phi(X)^\top, Y)$$

and

$$\begin{aligned} \mathrm{Ric}_\phi \circ d\phi(X)^\perp &= \sum_i \{D_X \nabla d\phi(E_i,E_i) - D_{E_i} \nabla d\phi(X,E_i)\} \\ &= \nabla_X \tau(\phi)^\perp - \mathrm{Trace}\ \nabla^2 d\phi(X)^\perp. \end{aligned}$$

<u>Theorem 1</u>. $\tau^V(\gamma_\phi) = \mathbb{C}(\text{Trace } \nabla^2 d\phi^\perp) = \mathbb{C}(mDH_\phi - Ric_\phi^\perp \circ d\phi)$.

Thus, γ_ϕ is a harmonic section of $\phi^{-1}G_m(N)$ iff Trace $\nabla^2 d\phi^\perp = 0$

iff $m\, DH_\phi = Ric_\phi^\perp \circ d\phi$.

<u>Proof</u>.

$$\tau^V(\gamma_\phi) = \sum_i {}^V\nabla_{E_i}(d\gamma_\phi)^V(E_i), \text{ for some orthonormal frame } \{E_i\} \text{ in } M.$$

$$= \sum_i \{{}^V\nabla_{E_i}((d\gamma_\phi)^V \circ E_i) - (d\gamma_\phi)^V(\nabla_{E_i}E_i)\}$$

$$= \sum_i \{{}^V\nabla_{E_i}\mathbb{C}(\nabla_{E_i}d\phi) - \mathbb{C}(\nabla d\phi(\nabla_{E_i}E_i)), \text{ by Proposition 4.}$$

$$= \sum_i \mathbb{C}\{D_{E_i}\nabla_{E_i} d\phi - \nabla d\phi(\nabla_{E_i}E_i)\}, \text{ by Corollary 1}$$

$$= \mathbb{C}\ (\text{Trace } \nabla^2 d\phi^\perp)$$

$$= \mathbb{C}(m\, DH_\phi - Ric_\phi^\perp \circ d\phi), \text{ by Lemma 3.}$$

Noting that $\mathbb{C}$ is an isomorphism on fibres, the theorem is proved. □

<u>Remark 5</u>. It is natural to expect the condition of the vanishing of only a component of Trace $\nabla^2 d\phi$. For, by the reduction theorem (Theorem 3.3.1), the vanishing of Trace $\nabla^2 d\phi$ is equivalent to ϕ being totally geodesic, and hence γ_ϕ being horizontal (Corollary 2).

<u>Remark 6.</u> Theorem 1 is in fact an application of *Codazzi's equation only* to Proposition 4 (cf. the last line of Remark 4; the proof of Theorem 1.1). Applying Gauss' equation will give us Theorem 2 below.

Lemma 4

If (N,h) has constant curvature "c", then $\mathrm{Ric}_\phi \circ d\phi = c(m-1)d\phi$.

Proof. The curvature tensor of (N,h) looks like ([K-N] vol 1, p. 203):

$$R^N(A,B)C = c\,\{h(C,B)A - h(C,A)B\}, \text{ for all } A,B,C \in \mathfrak{C}(TN).$$

Thus:

$$\mathrm{Ric}_\phi \circ d\phi(X) = \sum_i R^N(d\phi(X),\, d\phi(E_i))d\phi(E_i)$$

$$= c\,\sum_i\{h(d\phi(E_i),d\phi(E_i))d\phi(X)-h(d\phi(E_i),d\phi(X))d\phi(E_i)\}$$

$$= c\,\sum_i\{g(E_i,E_i)d\phi(X) - g(E_i,X)d\phi(E_i)\} = c(m-1)d\phi(X). \quad \square$$

Corollary 3.

If (N,h) has constant sectional curvature, then:

γ_ϕ is a harmonic section iff $DH_\phi = 0$. $\square$

Remark 7. In particular, Corollary 3 holds when $(N,h) = \mathbb{R}^n$, in which case the Gauss section is the graph of the Gauss map. Recalling Theorem 3.1.1, we recover the Ruh-Vilms Theorem.

Lemma 5.

If $n = m + 1$, then $\mathrm{Ric}_\phi^\perp \circ d\phi = (\mathrm{Ric}^N)^\perp \circ d\phi$.

Proof. Let $A \in \phi^{-1}(TN)$ be a unit normal, $X \in TM$, and $\{E_i\}$ an orthonormal frame in TM. Then:

$$h(\mathrm{Ric}_\phi \circ d\phi(X),A) = \sum_i h(R^N(d\phi(X),d\phi(E_i))d\phi(E_i),A)$$

$$= h(\mathrm{Ric}^N \circ d\phi(X),A) - h(R^N(d\phi(X),A)A,A)$$

since $\{d\phi(E_i),A\}_{i=1}^m$ is an orthonormal frame in TN.

$$= h(\mathrm{Ric}^N \circ d\phi(X),A), \text{ by skew-symmetry.} \quad \square$$

Corollary 4.

If $n = m+1$ and the Ricci curvature of (N,h) vanishes on $TM \times TM^\perp$ (in particular, when (N,h) is an *Einstein space*) then:

γ_ϕ is a harmonic section iff ϕ has constant mean curvature. $\square$

When $(N,h) = S^n$ we have the isomorphism
$$\begin{tikzcd} O(S^n) \arrow[rr, "\cong"] \arrow[dr] & & O(n+1) \arrow[dl] \\ & S^n & \end{tikzcd},$$
the fibering $O(n+1) \to S^n$ being (for example) projection of an orthogonal matrix onto its first column, and the action of $O(n)$ on $O(n+1)$ being right multiplication (where $O(n)$ is embedded as the subgroup $1 \times O(n)$ of $O(n+1)$). Then, $G_m(S^n) \cong O(n+1)/O(m) \times O(n-m)$, and the *canonical projection*

$$pr : O(n+1)/O(m) \times O(n-m) \to O(n+1)/O(m+1) \times O(n;m)$$

(namely, if $H < K < G$, then $pr : G/H \to G/K;\ gH \to gK$) is a map $pr : G_m(S^n) \to G_{m+1,n+1}$. The composition $pr \circ \gamma_\phi : M \to G_{m+1,n+1}$ is *Obata's Gauss map* ([Oba]). Writing k for the metric on $G_m(S^n)$, and k' for that on $G_{m+1,n+1}$, we have:

Lemma 6. $k^V = pr^*k'$.

Proof. We have the following fibrations:

$$\begin{tikzcd} O(n+1) \arrow[r, "\pi"] \arrow[d, "\xi"'] \arrow[dr, "\chi"] & G_{m+1,n+1} \\ S^n & G_m(S^n) \arrow[l, "\sigma"] \arrow[u, "pr"'] \end{tikzcd}$$

and the following identifications with subspace of $\mathfrak{o}(n+1)$:

$$T_{\chi(\mathbb{1})}G_mS^n = \left\{\begin{bmatrix} 0 & v_1 \cdots & v_n \\ -v_1 & \mathbb{0} & W \\ \vdots & & \\ -v_n & -W^t & \mathbb{0} \end{bmatrix} : v_i \in \mathbb{R},\ W \in M_{m\times(n-m)}(\mathbb{R})\right\}$$

$$\ker d\sigma(\chi(\mathbb{1})) = \left\{\begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & \mathbb{0} & W \\ 0 & -W^t & \mathbb{0} \end{bmatrix}\right\}$$

$$T_{\pi(\mathbb{1})}G_{m+1,n+1} = \left\{\begin{pmatrix} \mathbb{0} & W' \\ -W'^t & \mathbb{0} \end{pmatrix} : W' \in M_{(m+1)\times(n-m)}(\mathbb{R})\right\}$$

Then, $d\,pr\,(\chi(\mathbb{1}))$:
$$\begin{bmatrix} 0 & v_1 \cdots & v_n \\ -v_1 & \mathbb{0} & W \\ \vdots & & \\ -v_n & -W^t & \mathbb{0} \end{bmatrix} \to \begin{bmatrix} 0 \cdots 0 & v_{m+1} \cdots v_n \\ \vdots \quad\; \vdots & \\ 0 \cdots 0 & W \\ -v_{m+1} & \\ \vdots & \\ -v_n \quad -W^t & \mathbb{0} \end{bmatrix}$$

so that $\ker d\sigma(\chi(\mathbb{1})) \subset \ker d\,pr(\chi(\mathbb{1}))^{\perp}$. The result is now a consequence of pr being a Riemannian submersion. $\square$

Alternative Proof

Making the identifications $O(S^n) \cong O(n+1)$, $O(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times O(n+1)$, and $G_{m+1}(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times G_{m+1,n+1}$, the above fibrations give rise to the following morphisms:

$$
\begin{tikzcd}
 & G_m(S^n) \arrow[rr, "(i\circ\sigma{,}pr)"] \arrow[dd, dashed, "\sigma"] & & G_{m+1}(\mathbb{R}^{n+1}) \arrow[dd, bend left] \\
O(S^n) \arrow[ur, "\chi"] \arrow[rr, "(i\circ\xi{,}id)"] \arrow[dr, "\xi"'] & & O(\mathbb{R}^{n+1}) \arrow[ur, "(id{,}\pi)"'] \arrow[dr] & \\
 & S^n \arrow[rr, hook, "i"'] & & \mathbb{R}^{n+1}
\end{tikzcd}
$$

The isometric inclusion of $O(n)$ as the subgroup $1 \times O(n)$ of $O(n+1)$ factors through cosets to give an equivariant isometric embedding of $G_{m,n}$ in $G_{m+1,n+1}$ to which is associated the morphism $(i\circ\sigma,pr)$. Thus, $(i\circ\sigma,pr)$ is a fibre isometry, whence:

$$k^V = (i\circ\sigma,pr)^* \left(\sum_i dx_i^2 \times k'\right)^V = pr^*k'$$

where the product $\sum_i dx_i^2 \times k'$ describes the metric on $G_{m+1}(\mathbb{R}^{n+1})$ □

Obata defines the third fundamental form of a Riemannian immersion into a space form as the pullback of k' by the Obata Gauss map. In view of Lemma 6, we may generalize this by defining the *third fundamental form* of an arbitrary Riemannian immersion $\phi:(M,g) \to (N,h)$ to be the quadratic differential $\gamma_\phi^* k^V$. Then Corollary 2 may be restated as:

<u>Corollary 5</u>.

ϕ has vanishing third fundamental form iff ϕ is totally geodesic. □

By an application of the Gauss equation to the identity of Proposition 4, we obtain the following identity of quadratic differentials on M:

<u>Theorem 2</u>.

$$\mathrm{Ric}^M - h(\tau(\phi),\nabla d\phi) + \gamma_\phi{}^*k^V = \mathrm{Ric}_\phi \circ d\phi.$$

<u>Proof</u>. By Proposition 4, and the fact that $\mathbb{C}$ is isometric (Proposition 3):

$$\gamma_\phi{}^*k^V(X,Y) = k((d\gamma_\phi)^V(X),(d\gamma_\phi)^V(Y)) = k(\mathbb{C}(\nabla_X d\phi),\mathbb{C}(\nabla_Y d\phi))$$

$$= \langle\nabla_X d\phi,\nabla_Y d\phi\rangle$$

$$= \sum_i h(\nabla d\phi(X,E_i),\nabla d\phi(Y,E_i)), \text{ for some orthonormal frame } \{E_i\}$$

$$= \sum_i g(B(X,E_i)E_i,Y) + h(\tau(\phi),\nabla d\phi(X,Y)), \text{ by definition of } B$$

(cf. Remark 4)

$$= \sum_i \{h(R^N(d\phi(X),d\phi(E_i))d\phi(E_i),d\phi(Y)) - g(R^M(X,E_i)E_i,Y)\}$$

$$+ h(\tau(\phi),\nabla d\phi(X,Y)), \text{ by the equation of Gauss}$$

$$= \mathrm{Ric}_\phi\,(d\phi(X),d\phi(Y)) - \mathrm{Ric}^M(X,Y) + h(\tau(\phi),\nabla d\phi(X,Y)). \quad \square$$

<u>Remark 8</u>. This generalizes the formula proved in [Oba] for the case where (N,h) has constant curvature "c":

$$\psi - II_N + III = c(m-1)ds^2$$

where $\psi = \mathrm{Ric}^M$, $II_N = h(\tau(\phi), \nabla d\phi)$, III is the 3^{rd} fundamental form of ϕ, and $c(m-1)ds^2 = \mathrm{Ric}_\phi \circ d\phi$ (Lemma 4).

By an *Einsteinian* immersion we understand a Riemannian immersion ϕ into (N,h) with $\mathrm{Ric}_\phi \circ d\phi$ a (not necessarily constant) scalar multiple of ϕ^*h. In particular, this includes the case of (N,h) a space form, and by the Weitzenböck formula:

Proposition 5.

If ϕ is an Einsteinian immersion, then γ_ϕ is a harmonic section iff $DH_\phi = 0$. □

As a corollary of Theorem 2, we have:

Corollary 6.

Any three of the following conditions imply the fourth:

(i) ϕ is Einsteinian.

(ii) ϕ has *vertically conformal* Gauss section.

(iii) ϕ is *pseudo-umbilical* (i.e. $h(\tau(\phi), \nabla d\phi)$ is proportional to ϕ^*h).

(iv) (M, ϕ^*h) is an Einstein space. □

§3. THE GAUSS SECTION OF A RIEMANNIAN FOLIATION

We firstly review some terminology and elementary geometrical properties of foliations (cf [KT2], [Rein]). Suppose that Δ is an m-dimensional foliation of N, with algebraic normal bundle $Q = TN/\Delta$, and let $\pi: TN \to Q$ be the quotient morphism. There is a natural Δ-partial connection in Q (the "*Bott connection*") with (partial) covariant derivative:

$$\mathring{\underline{D}}_X \xi = \pi[X, Y_\xi]$$

for any $X \in \mathfrak{C}(\Delta)$, $\xi \in \mathfrak{C}(Q)$, and $Y_\xi \in \mathfrak{C}(TN)$ with $\pi(Y_\xi) = \xi$.

Remark 1. The integrability of Δ is essential for $\overset{\circ}{\underline{D}}$ to be well-defined.

Introducing a Riemannian metric h on N gives a geometric normal bundle $\Delta^\perp \subset TN$, and hence a splitting of the short exact sequence of vector bundles:

$$0 \to \Delta \to TN \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \to 0, \text{ where } \sigma(Q) = \Delta^\perp.$$

This has the following consequences:

(i) σ pulls back h to a fibre metric $\underline{h}$ in Q.

(ii) The partial Bott connection in Q can be completed, by defining:

$$\underline{D}_X\xi = \pi(\nabla_X(\sigma\xi)), \text{ for any } X \in \mathfrak{C}(\Delta^\perp),\ \xi \in \mathfrak{C}(Q).$$

The metric h is said to be *bundle-like for* Δ, and Δ a *Riemannian foliation* of (N,h), if $(Q,\underline{D},\underline{h})$ is a Riemannian vector bundle.

Lemma 1.

h is bundle-like for Δ iff $h(\nabla_Y X, Z) + h(Y, \nabla_Z X) = 0$ for all $X \in \mathfrak{C}(\Delta)$ and $Y,Z \in \mathfrak{C}(\Delta^\perp)$.

Proof. h is bundle-like for Δ iff $X.\underline{h}(\xi,\nu) = \underline{h}(\underline{D}_X\xi,\nu) + \underline{h}(\xi,\underline{D}_X\nu)$ for all $X \in \mathfrak{C}(TN)$ and $\xi,\nu \in \mathfrak{C}(Q)$.

iff $X.\underline{h}(\xi,\nu) = \begin{cases} \underline{h}(\pi[X,\sigma\xi],\nu) + \underline{h}(\xi,\pi[X,\sigma\nu]), & \text{whenever } X \in \mathfrak{C}(\Delta) \\ \underline{h}(\pi(\nabla_X\sigma\xi),\nu) + \underline{h}(\xi,\pi(\nabla_X\sigma\nu)), & \text{whenever } X \in \mathfrak{C}(\Delta^\perp) \end{cases}$

$$\text{iff } X.h(Y,Z) = \begin{cases} h([X,Y],Z) + h(Y,[X,Z]), & \text{whenever } X \in \mathfrak{C}(\Delta) \\ h(\nabla_X Y,Z) + h(Y,\nabla_X Z), & \text{whenever } X \in \mathfrak{C}(\Delta^{\perp}) \end{cases}$$

$$= h([X,Y],Z) + h(Y,[X,Z]), \text{ for all } X \in \mathfrak{C}(\Delta),\ Y,Z \in \mathfrak{C}(\Delta^{\perp})$$

since ∇ is metric. The result follows by comparing with the equation $\nabla h = 0$, and using the symmetry of ∇. □

As an example, we cite:

Proposition 1

Let $\phi:(N,h) \to (M,g)$ be a *Riemannian submersion*, and $\Delta = \ker d\phi$. Then h is bundle-like for Δ.

Proof. Let $X \in \mathfrak{C}(\Delta)$ and $Y,Z \in \mathfrak{C}(\Delta^{\perp})$. Then:

$$h(\nabla_Y X,Z) + h(Y,\nabla_Z X) = g(d\phi(\nabla_Y X),d\phi(Z)) + g(d\phi(Y),d\phi(\nabla_Z X))$$

since Y,Z are horizontal.

$$= g(\nabla_Y(d\phi\circ X)-\nabla_Y d\phi(X),d\phi(Z)) + g(d\phi(Y),\nabla_Z(d\phi\circ X) - \nabla_Z d\phi(X))$$

$$= -g(\nabla_Y d\phi(X),d\phi(Z))-g(d\phi(Y),\nabla_Z d\phi(X)), \text{ since } X \text{ is vertical.}$$

$$= -g(\nabla_X d\phi(Y),d\phi(Z))-g(d\phi(Y),\nabla_X d\phi(Z)), \text{ since } \nabla d\phi \text{ is symmetric.}$$

$$= g(d\phi(\nabla_X Y),d\phi(Z)) + g(d\phi(Y),d\phi(\nabla_X Z)$$

$$-g(\nabla_X(d\phi\circ Y),d\phi(Z)) - g(d\phi(Y),\nabla_X(d\phi\circ Z))$$

$$= h(\nabla_X Y,Z) + h(Y,\nabla_X Z) - X.g(d\phi(Y),d\phi(Z))$$

$= X.h(Y,Z) - X.h(Y,Z)$, since Y,Z are horizontal, and $(\phi^{-1}TN,\phi^{-1}\nabla,\phi^{-1}g)$ a Riemannian vector bundle (Remark 1.2.2)

$= 0$, so that, by Lemma 1, h is bundle-like for ∇. □

Given any metric h (not necessarily bundle-like), say that a distribution Δ is h-*parallel* whenever:

$$\gamma'(0) \perp \Delta_{\gamma(0)} \Rightarrow \gamma'(t) \perp \Delta_{\gamma(t)}$$

for any geodesic $\gamma(t)$ of (N,h). If in addition Δ is integrable, say that Δ has *parallel leaves*.

Remark 2. Given an affine connection ∇ on N, the distribution Δ is said to be ∇-*parallel* if Δ is invariant under the holonomy of ∇. If ∇ is torsion-free, a ∇-parallel distribution is necessarily integrable (cf [Wil]).

Proposition 2.

If h is bundle-like for a foliation Δ, then Δ has parallel leaves.

Proof. (As an alternative, see [Rein] Proposition 2).

Let $\gamma(t)$ be a geodesic of (N,h) with $\gamma'(0) \perp \Delta_{\gamma(0)}$, and let $\{E_i^t\}_{i=1}^m$ be an orthonormal frame at $\gamma(t)$ spanning $\Delta_{\gamma(t)}$. Using the Riemannian connection of the metric induced in the leaf through $\gamma(t)$, form the normal coordinates determined by $\{E_i^t\}$.

$$\frac{d}{dt}\, h(E_i^t,\gamma'(t)) = h(\nabla_{\frac{d}{dt}} E_i^t,\gamma'(t)) + h(E_i^t,\nabla_{\frac{d}{dt}}\gamma')$$

(working in $\gamma^{-1}(TN)$)

$$= h(\nabla_{\gamma'(t)}E_i^t,\gamma'(t)), \text{ since } \gamma(t) \text{ is a geodesic.}$$

$$= h(\nabla_{\gamma'(t)^T} E_i^t,\gamma'(t)^T) + h(\nabla_{\gamma'(t)^T}E_i^t,\gamma'(t)^\perp) + h(\nabla_{\gamma'(t)^\perp}E_i^t,\gamma'(t)^T)$$

since, by Lemma 1, $h(\nabla_{\gamma'(t)^\perp} E_i^t,\gamma'(t)^\perp) = 0$.

Applying Gauss's formula ([K-N] vol 2, p. 15) to the first two summands:

$h(\nabla_{\gamma'(t)^T} E_i^t, \gamma'(t)^T) = h(\tilde{\nabla}_{\gamma'(t)^T} E_i^t, \gamma'(t)^T) = 0$, by normality at $\gamma(t)$ where $\tilde{\nabla}$ denotes covariant differentiation in the leaf.

$h(\nabla_{\gamma'(t)^T} E_i^t, \gamma'(t)^\perp) = h(\alpha(\gamma'(t)^T, E_i^t), \gamma'(t)^\perp)$, where α is the second fundamental form of the leaf.

Extent $\gamma'(t)^\perp$ to a local field Y along the leaf through $\gamma(t)$, such that $[Y, E_i^t]_{\gamma(t)} = 0$. Then, by Weingarten's formula:

$$h(\nabla_{\gamma'(t)^\perp} E_i^t, \gamma'(t)^T) = h(\nabla_{E_i^t} Y, \gamma'(t)^T) = -h(A_Y E_i^t, \gamma'(t)^T)$$

where A is the shape operator of the leaf.

$$= -h(\alpha(E_i^t, \gamma'(t)^T), Y) \quad \text{(cf [K-N] vol 2, p. 14).}$$

Thus, by the symmetry of α, $\frac{d}{dt} h(E_i^t, \gamma'(t)) = 0$, so that

$$h(E_i^t, \gamma'(t)) = \text{const.} = h(E_i^o, \gamma'(0)) = 0. \quad \square$$

A converse to Proposition 2 exists in codimension 1:

<u>Proposition 3</u>.

If Δ is a codimension 1 foliation of (N,h) with parallel leaves, then h is bundle-like for Δ.

<u>Proof</u>. Let $Y \in \Delta^\perp$ and $X \in \mathfrak{C}(\Delta)$. If $\gamma(t)$ is the geodesic with initial velocity Y, then $h((\tau_N)_t^\gamma(X), \gamma'(t)) = 0$, since parallel translation along a geodesic preserves angles. Now, because Δ has parallel leaves:

$$h(\Delta_{\gamma(t)}, \gamma'(t)) = 0.$$

Thus, in codimension 1, $(\tau_N)^\gamma_t(X) \in \Delta_{\gamma(t)}$; equivalently, $(\nabla_Y X)^\perp = 0$. So, by Lemma 1, h is bundle-like for Δ. □

We may think of an m-plane distribution Δ as a section of the Grassmann bundle $\sigma: G_m N \to N$, the *Gauss section* of the distribution. Concerning its vertical differential:

Lemma 2.

If Δ is a codimension-1 Riemannian foliation of (N,h), then $(d\Delta)^V \restriction \Delta^\perp = 0$.

Proof. Let $\gamma(t)$ be the geodesic determined by $Y \in \Delta^\perp$. Recalling Proposition 2.2.1:

$$(d\Delta)^V(Y) = \frac{d}{dt}\Big|_0 (\Delta\circ\gamma)^V(t) = \frac{d}{dt}\Big|_0 (\sigma^\gamma_t)^{-1}(\Delta\circ\gamma(t)).$$

By Lemma 2.2, $X \in (\sigma^\gamma_t)^{-1}(\Delta\circ\gamma(t))$ iff $(\tau^\gamma_N)_t(X) \in \Delta \circ \gamma(t)$.

Now, since Δ has parallel leaves (Proposition 2) and is of codimension 1

$$(\tau_N)^\gamma_t : \Delta_{\gamma(0)} \to \Delta_{\gamma(t)}$$ (because parallel translation along geodesics preserves angles).

Thus $(\sigma^\gamma_t)^{-1}(\Delta\circ\gamma(t)) = \Delta\circ\gamma(0)$, so that $(d\Delta)^V(Y) = 0$. □

Remark 3. The proof of Lemma 3 rests on the fact that, in codimension 1, parallel translation along orthogonal geodesics preserves the distribution. In higher codimension this is no longer necessarily true. For example, if $\phi:(N,h) \to (M,g)$ is a Riemannian submersion, so that ker $d\phi$ is a Riemannian foliation of (N,h) (Proposition 1), then: Lemma 2

$(TN)^V$ is preserved by parallel translation along horizontal geodesics

iff $(\nabla_Y X)^{\perp} = 0$, for all vertical X and horizontal Y

iff $\nabla d\phi((TN)^V,(TN)^H) = 0$, since $\nabla_Y d\phi(X) = \nabla_Y(d\phi\circ X) - d\phi(\nabla_Y X)$

$$= -d\phi(\nabla_Y X)$$

iff $(TN)^H$ is integrable, by Lemma 3.2 of [Vil].

Denote by $\phi_y : M_y \to N$ the embedding of the leaf through $y \in N$. The *mean curvature* H_Δ of Δ is then defined:

$$H_\Delta(y) = H_{\phi_y}(y).$$

Also, $\Delta\circ\phi_y = \gamma_{\phi_y}$, the Gauss section of M_y. By piecing together the vertical tension fields of the γ_{ϕ_y}, we are able to extend Theorem 2.1 to codimension 1 Riemannian foliations:

Theorem 1.

If Δ is a codimension 1 Riemannian foliation of (N,h), then:

Δ is harmonic section of $G_{n-1}N$ iff $\mathring{D}H_\Delta = \frac{1}{n-1}\,(\mathrm{Ric}^N)^{\perp}$

where $\mathring{D}$ denotes partial covariant differentiation in $\Delta^{\perp}$ along Δ.

Proof. Since $\Delta\circ\phi_y = \gamma_{\phi_y}$:

$$\mathrm{Trace}\ {}^V\nabla(d\Delta)^V(d\phi_y,d\phi_y) = {}^V\tau(\gamma_{\phi_y}) - (d\Delta)^V\tau(\phi_y).$$

By Corollary 2.5.1 (i), $\tau(\phi_y)$ is normal to M_y, so that $(d\Delta)^V\tau(\phi_y) = 0$, by Lemma 2. Again by Lemma 2, $\mathrm{Trace}\ {}^V\nabla(d\Delta)^V(d\phi_y,d\phi_y) = {}^V\tau(\Delta)$. Thus, by Theorem 2.1 and Lemma 2.5:

Δ is a harmonic section iff each γ_{ϕ_y} is a harmonic section

iff $DH_{\phi_y} = \frac{1}{n-1} (\mathrm{Ric}^N)^{\perp} \circ d\phi_y$, for each $y \in N$. □

Plugging in Corollary 2.4:

Corollary

If, in addition to the hypotheses of Theorem 1, the Ricci curvature of (N,h) vanishes on $\Delta \times \Delta^{\perp}$ (in particular, if (N,h) is an Einstein space), then:

Δ is a harmonic section iff the leaves of Δ all have constant mean curvature. □

Example. Foliation by isoparametric hypersurfaces

The notion of an isoparametric family of hypersurfaces in a space form originated with E. Cartan and has been studied more recently by K. Nomizu ([Nom]). If (N,h) is a space form, a function $f:N \to \mathbb{R}$ is said to be *isoparametric* if:

(i) $e(f) = \Phi_1 \circ f$ ($e(f) = \frac{1}{2} \|\mathrm{grad}\ f\|^2$)

(ii) $\tau(f) = \Phi_2 \circ f$ ($\tau(f)$ is just the Laplacian of f),

where $\Phi_1, \Phi_2 : \mathbb{R} \to \mathbb{R}$ are smooth reparametrisations of f. If t is a regular value of f, $f^{-1}(t)$ is called an *isoparametric hypersurface;* otherwise, $f^{-1}(t)$ is called a *focal variety.* Among the properties of isoparametric functions are the following:

(a) The gradient flow of an isoparametric function is along (normal) geodesics ([Nom]) Proposition 2).

(b) The principal curvatures of an isoparametric hypersurface are constant ([Nom] §2).

(c) An isoparametric function has at most two focal varieties ([Car]), each of which is a minimal submanifold ([Nom]).

Let K denote the union of focal varieties. Then, by (c), K is a closed, nowhere dense subset of N, so that $N \setminus K$ is an open dense submanifold of N, with a foliation Δ by isoparametric hypersurfaces. Property (a) then says that Δ has parallel leaves so that, by Proposition 2, Δ is a Riemannian foliation of (N,h). Since (N,h) is certainly Einstein, the Corollary may be used to deduce:

Theorem 2.

Let (N,h) be a space form, and let Δ be the foliation of $N \setminus K$ by an isoparametric family of hypersurfaces. Then, Δ is a harmonic section of $G_{n-1}(N \setminus K)$.

Proof. Property (b) implies that each leaf has constant mean curvature. □

Remark 4. In his thesis, P. Baird has provided a plentiful supply of examples of isoparametric functions, and has also given the natural generalisation of the idea to include cases when (N,h) is no longer a space form. ([Bai]). Theorem 2 may then be extended to produce harmonic sections from foliations of Einstein spaces by *generalised* families of isoparametric hypersurfaces.

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